Theorem. *For $n = p\,q$ where p, q are distinct odd primes, the number of Euler liars in $\mathbb{Z}_n^*$ equals $\frac{1}{2}d^2$, where $d = \gcd(p-1, q-1)$.*

Proof. Write $p - 1 = a\,d$ and $q - 1 = b\,d$. Note that $n - 1 = C\,d$ where $C = a + b + a\,b\,d$, so $p - 1 \mid a\,(n-1)$ and $q - 1 \mid b\,(n-1)$. One might say that composite squarefree integers with only two prime factors come sort of close to Korselt's criterion for Carmichael numbers, failing only by factors a, b for the two prime factors p, q respectively. Note that a and b are coprime, C is coprime to both a and b, and

$$\frac{a\,(n-1)}{p-1} = \frac{b\,(n-1)}{q-1} = C. \tag{1}$$

If $x \in \mathbb{Z}_n^*$ is an Euler liar then $x^{n-1} \equiv 1 \pmod{n}$. Let g_p be a primitive root $\pmod{p}$, and g_q one $\pmod{q}$. Let $A_p, A_q \in \mathbb{Z}_n^*$ satisfy $A_p \equiv g_p \pmod{p}$, $A_p \equiv 1 \pmod{q}$, $A_q \equiv 1 \pmod{p}$, $A_q \equiv g_q \pmod{q}$. Then every $x \in \mathbb{Z}_n^*$ can be written as $x \equiv A_p^{e_p} A_q^{e_q} \pmod{n}$ for a unique $(e_p, e_q) \in \mathbb{Z}_{p-1} \times \mathbb{Z}_{q-1}$. Because $\mathrm{ord}_n(A_p) = p - 1$ and $\mathrm{ord}_n(A_q) = q - 1$ we have that

$$\begin{aligned}
x^{n-1} \equiv 1 \pmod{n} \quad & \text{if and only if} \quad p-1 \mid e_p\,(n-1) \text{ and } q-1 \mid e_q\,(n-1) \\
& \text{if and only if} \quad a\,d \mid e_p\,(n-1) \text{ and } b\,d \mid e_q\,(n-1) \\
& \text{if and only if} \quad a \mid e_p\,C \text{ and } b \mid e_q\,C \\
& \text{if and only if} \quad a \mid e_p \text{ and } b \mid e_q,
\end{aligned}$$

because, as we saw abov,e C is coprime to a and b. So we have $e_p = a\,f_p$ and $e_q = b\,f_q$, where $f_p, f_q \in \{0, 1, \ldots, d-1\}$, because $\dfrac{p-1}{a} = \dfrac{q-1}{b} = d$. Clearly there are exactly $\dfrac{\phi(n)}{a\,b} = d^2$ elements $x \in \mathbb{Z}_n^*$ with $x^{n-1} \equiv 1 \pmod{n}$.

It follows that $x^{(n-1)/2} \equiv A_p^{a\,(n-1)\,f_p/2} A_q^{b\,(n-1)\,f_q/2} \pmod{n}$, so by (1)

$$x^{(n-1)/2} \equiv A_p^{(p-1)/2\,f_p\,C} A_q^{(q-1)/2\,f_q\,C} \pmod{n}. \tag{2}$$

Clearly $\left(\dfrac{A_p}{n}\right) = \left(\dfrac{A_q}{n}\right) = -1$, so $\left(\dfrac{x}{n}\right) = (-1)^{e_p+e_q}$, so

$$\left(\frac{x}{n}\right) = (-1)^{a\,f_p + b\,f_q}. \tag{3}$$

We now distinguish two cases: a and b are both odd, or one of them is even.

If a and b are both odd, then C is even, and from (2) we get $A_p^{(p-1)/2\,f_p\,C} \equiv 1 \pmod{n}$ and $A_q^{(q-1)/2\,f_q\,C} \equiv 1 \pmod{n}$, so $x^{(n-1)/2} \equiv 1 \pmod{n}$, independent of f_p, f_q. But from (3) $\left(\dfrac{x}{n}\right) = (-1)^{a\,f_p + b\,f_q} = (-1)^{f_p+f_q}$, and this is equally often $+1$ as -1. This shows that in this case the number of Euler liars is $\frac{1}{2}d^2$.

If one of a, b is even, then C is odd, and the situation is different. There are four square roots of 1 in $\mathbb{Z}_n^*$, namely ± 1 and $\pm\alpha$ where $\alpha \equiv A_p^{(p-1)/2} \equiv -A_q^{(q-1)/2} \pmod{n}$. With this shorthand notation we find from (2) $x^{(n-1)/2} \equiv (-1)^{f_q\,C}\,\alpha^{(f_p+f_q)\,C} \equiv (-1)^{f_q}\,\alpha^{f_p+f_q} \pmod{n}$. When $f_p + f_q$ is odd we get $x^{(n-1)/2} \equiv \pm\alpha \not\equiv \pm 1 \pmod{n}$, so then x is an Euler witness. And when $f_p + f_q$ is even then we get $x^{(n-1)/2} \equiv (-1)^{f_q} \pmod{n}$. But from (3) using $f_p \equiv f_q \pmod{2}$ we get $\left(\dfrac{x}{n}\right) = (-1)^{(a+b)\,f_q} = (-1)^{f_q}$, so then x is an Euler liar. Both possibilities for $f_p + f_q$ occur equally often because $f_p, f_q \in \{0, 1, \ldots, d-1\}$ and d is even. This shows that also in this case the number of Euler liars is $\frac{1}{2}d^2$. □