ON THE DIOPHANTINE EQUATIONS $x^2+74=y^5$ AND $x^2+86=y^5$

by MAURICE MIGNOTTE and BENJAMIN M. M. DE WEGER

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Abstract. J. H. E. Cohn solved the diophantine equations $x^2+74=y^n$ and $x^2+86=y^n$, with the condition $5 \nmid n$, by more or less elementary methods. We complete this work by solving these equations for $5 \mid n$, by less elementary methods.

1. Introduction. In a recent paper [**2**], J. H. E. Cohn considered the diophantine equations

$$x^2+C=y^n, \tag{1}$$

in positive integers x, y, n with $n \geq 2$, for the positive integers $C \leq 100$. He was able to solve 77 of them, and among the ones he did not complete are the two cases $C=74$ and $C=86$, which we consider here.

More precisely, Cohn proved that for $5 \nmid n$ there is only the solution $x=985$, $y=99$, $n=3$ for $C=74$, and no solution for $C=86$, Other solutions obviously may only occur for the equation

$$x^2+C=y^5. \tag{2}$$

The reason why Cohn's method fails for equation (2) in the cases $C=74, 86$ is that the class number of the field $\mathbb{K}=\mathbb{Q}(\sqrt{-C})$ in these cases is a multiple of 5.

In this paper we study the diophantine equation (2) for $C=74$ and $C=86$, and prove the following result.

THEOREM 1. *Equation* (2) *in positive integers* x, y *has only the solution* $x=13$, $y=3$ *in the case* $C=74$, *and has no solution in the case* $C=86$.

Combined with Cohn's results of [**2**], Theorem 1 immediately implies the following result.

THEOREM 2. *Equation* (1) *in positive integers* x, y, n *with* $n \geq 2$ *has only the solutions* $x=13$, $y=3$, $n=5$ *and* $x=985$, $y=99$, $n=3$ *in the case* $C=74$, *and has no solution in the case* $C=86$.

Our proof is based on diophantine approximation theory. First we reduce equation (2) to a set of quintic Thue equations. Then, following classical arguments as outlined in [**3**], and using the theory of linear forms in logarithms of algebraic numbers as in [**1**], we derive large upper bounds for the unknowns in these Thue equations. Finally, by computational diophantine approximation techniques, following [**3**], and using a new idea of Yuri Bilu to improve efficiency, we reduce these large upper bounds to small upper bounds, and thus are able to find all the solutions.

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2. Thue equations. We consider equation (2) as an equation of ideals in the field $\mathbb{K} = \mathbb{Q}(\sqrt{-C})$. Let $\mathfrak{b}$ be the fifth-power-free part of $\langle x + \sqrt{-C}\rangle$. Then we can write

$$\langle x + \sqrt{-C}\rangle = \mathfrak{b}\mathfrak{a}^5 \tag{3}$$

for some integral ideal $\mathfrak{a}$ in $\mathbb{K}$. Multiplying by the conjugate equation we see that (2) leads to the observation that $\mathfrak{b}\bar{\mathfrak{b}}$ is a fifth power. From this it's easy to conclude that $\mathfrak{b} = \langle 1\rangle$, so that (3) becomes

$$\langle x + \sqrt{-C}\rangle = \mathfrak{a}^5. \tag{4}$$

In both the cases $C = 74, 86$ the class group of $\mathbb{K}$ is cyclic of order 10, and the prime 3 splits, say as

$$\langle 3\rangle = \mathfrak{p}\bar{\mathfrak{p}}.$$

The ideal class of $\mathfrak{p}$ has order 5 resp. 10 in the class group, in case $C = 74$ resp. $C = 86$. When we put

$$\mathfrak{q} = \mathfrak{p} \quad \text{if} \quad C = 74, \qquad \mathfrak{q} = \mathfrak{p}^2 \quad \text{if} \quad C = 86,$$

then it is clear that there exists an integer k with $|k| \le 2$ such that $\mathfrak{q}^{-k}\mathfrak{a}$ is principal, and there exist $u, v \in (N\mathfrak{q})^{-\max\{0,k\}}\mathbb{Z}$ such that

$$\mathfrak{q}^{-k}\mathfrak{a} = \langle u + v\sqrt{-C}\rangle.$$

Put

$$U = u^5 - 10Cu^3v^2 + 5C^2uv^4, \qquad V = 5u^4v - 10Cu^2v^3 + C^2v^5; \tag{5}$$

then $(u + v\sqrt{-C})^5 = U + V\sqrt{-C}$. It now follows by (4) that

$$x + \sqrt{-C} = \gamma^k(U + V\sqrt{-C}), \tag{6}$$

where γ is a generator of the principal ideal $\mathfrak{q}^5$. In fact, without loss of generality, in the case $C = 74$ we may take $\gamma = 13 + \sqrt{-74}$, and in the case $C = 86$ we may take $\gamma = 157 + 20\sqrt{-86}$. Comparing the imaginary parts in equation (6) and multiplying by a common demoninator (which is a power of 3) leads to

$$\alpha U + \beta V = m, \tag{7}$$

where the α, β, m are as in the following table.

	$C = 74$			$C = 86$		
k	α	β	m	α	β	m
0	0	1	1	0	1	1
± 1	1	± 13	3^5	20	± 157	3^{10}
± 2	26	± 95	3^{10}	6280	∓ 9751	3^{20}

Note that if $k = 0$ then (7) reads $V = 1$, and by $V = v(5u^4 - 10Cu^2v^2 + C^2v^4)$ this case is trivial: there are no solutions in both cases. And the cases with $k < 0$ reduce to the corresponding cases with $k > 0$ on changing the sign of V. Hence from now on we assume $k = 1$ or $k = 2$.

We substitute (5) into equation (7), and thus obtain a quintic Thue equation

$$f_0u^5+f_1u^4v+f_2u^3v^2+f_3u^2v^3+f_4uv^4+f_5v^5=m, \tag{8}$$

with parameters as in the following table.

C	k	f_0	f_1	f_2	f_3	f_4	f_5	m
74	1	1	65	−740	−9620	27 380	71 188	3^5
74	2	26	475	−19 240	−70 300	711 880	520 220	3^{10}
86	1	20	785	−17 200	−135 020	739 600	1 161 172	3^{10}
86	2	6280	−48 755	−5 400 800	8 385 860	232 234 400	−72 118 396	3^{20}

Observe that the Thue equation (8) for $C=74$, $k=2$ is impossible modulo 11. For the two Thue equations (8) in the case $C=86$, which we will prove to have no solutions, we did not find a prime p such that these equations are impossible modulo p.

We make some further simplifications to the equations (8).

In the case $C=74$, $k=1$ we observe that $3\mid(u+v)$. Therefore we put $X=\frac{1}{3}(u+v)$, $Y=v$, and thus obtain the Thue equation

$$X^5+20X^4Y-110X^3Y^2-260X^2Y^3+545XY^4+144Y^5=1. \tag{9}$$

Below we prove that $X=1$, $Y=0$ is the only solution of (9). That suffices to prove Theorem 1 for the case $C=74$.

In the case $C=86$, $k=1$ we observe that $9\mid(u+2v)$. We now put $X=\frac{1}{9}(u+2v)$, $Y=v$, and thus obtain the Thue equation

$$20X^5+65X^4Y-280X^3Y^2-20X^2Y^3+160XY^4-12Y^5=1. \tag{10}$$

In the case $C=86$, $k=2$ we observe that $81\mid(4u-v)$. We put $X=\frac{1}{81}(-4u+v)$, $Y=\frac{1}{81}(-u-20v)$, and thus obtain the Thue equation

$$4X^5-80X^4Y+100X^3Y^2+320X^2Y^3-355XY^4+4Y^5=1. \tag{11}$$

Below we prove that (10) and (11) have no solutions. That suffices to prove Theorem 1 for the case $C=86$.

3. Quintic fields. Let's now study the quintic fields associated to the Thue equations (9), (10) and (11). We used Pari 1.38 to compute the results in this section, and checked them by Kant 2.

In the case of equation (9) (i.e. $C=74$, $k=1$), we work in $\mathbb{L}=\mathbb{Q}(\theta)$, where θ satisfies

$$\theta^5-15\theta^3+45\theta-26=0.$$

This field is totally real, its discriminant is $273\,800\,000=2^65^537^2$, a basis for the ring of integers is $\{1,\theta,\theta^2,\theta^3,\theta^4\}$, the class group is trivial, and a set of fundamental units is given by

$$\epsilon_1=-3-\theta, \qquad \epsilon_2=75-18\theta-29\theta^2+\theta^3+2\theta^4,$$

$$\epsilon_3=-1-4\theta+9\theta^2-\theta^3-2\theta^4, \qquad \epsilon_4=-19+23\theta+16\theta^2-10\theta^3-4\theta^4,$$

so that the regulator is 1386.37307 .... The Galois group is M(20), the metacyclic group of degree 5 generated by the permutations (12345) and (2354).

We put

$$\psi = -4 + 6\theta - \theta^3.$$

This algebraic integer satisfies

$$\psi^5 + 20\psi^4 - 110\psi^3 - 260\psi^2 + 545\psi + 144 = 0.$$

Hence the Thue equation (9) can be written as the norm form equation $N_{\mathbb{L}/\mathbb{Q}}(X - Y\psi) = 1$, leading (neglecting a sign, without loss of generality) to

$$X - Y\psi = \epsilon_1^{a_1}\epsilon_2^{a_2}\epsilon_3^{a_3}\epsilon_4^{a_4}. \tag{12}$$

In the cases of equations (10) and (11) (i.e. $C = 86$, $k = 1, 2$), we work in $\mathbb{L} = \mathbb{Q}(\theta)$, where θ satisfies

$$\theta^5 - 30\theta^3 + 180\theta - 160 = 0.$$

This field is totally real, its discriminant is $369\,800\,000 = 2^6 5^5 43^2$, a basis for the ring of integers is $\{1, \theta, \frac{1}{2}\theta^2, \frac{1}{2}\theta^3, \frac{1}{4}\theta^4\}$, the class group is trivial, and a set of fundamental units is given by

$$\epsilon_1 = 31 - 8\theta - 6\theta^2 + \tfrac{1}{2}\theta^3 + \tfrac{1}{4}\theta^4, \qquad \epsilon_2 = -9 + 10\theta + \theta^2 - \tfrac{3}{2}\theta^3 - \tfrac{1}{4}\theta^4,$$

$$\epsilon_3 = 81 - 49\theta - \tfrac{51}{2}\theta^2 + 2\theta^3 + \theta^4, \qquad \epsilon_4 = -209 + 55\theta + 43\theta^2 - 2\theta^3 - \tfrac{3}{2}\theta^4,$$

so that the regulator is 1522.07808 .... The Galois group is again M(20). The prime 2 splits as $\langle 2\rangle = \mathfrak{p}\mathfrak{q}^2$, where $N\mathfrak{p} = 2$, $N\mathfrak{q} = 4$, and the prime 5 ramifies completely as $\langle 5\rangle = \mathfrak{r}^5$.

For the case of equation (10), (i.e. $C = 86$, $k = 1$), we put

$$\psi = \tfrac{19}{4} - \theta - \tfrac{9}{8}\theta^2 + \tfrac{3}{80}\theta^4.$$

This algebraic number satisfies

$$20\psi^5 + 65\psi^4 - 280\psi^3 - 20\psi^2 + 160\psi - 12 = 0,$$

and $\mathfrak{p}^2\mathfrak{r}$ is the denominator of the ideal $\langle\psi\rangle$. Hence the Thue equation (10) can be written as the norm form equation $20N_{\mathbb{L}/\mathbb{Q}}(X - Y\psi) = 1$, leading (neglecting a sign, without loss of generality) to

$$X - Y\psi = \mu\epsilon_1^{a_1}\epsilon_2^{a_2}\epsilon_3^{a_3}\epsilon_4^{a_4}, \tag{13}$$

were $\langle\mu\rangle = \mathfrak{p}^{-2}\mathfrak{r}^{-1}$. In fact, we can take

$$\mu = -\tfrac{5}{4} + \tfrac{3}{8}\theta^2 - \tfrac{1}{80}\theta^4.$$

For the case of equation (11) (i.e. $C = 86$, $k = 2$), we put

$$\psi = 4 - 3\theta + \tfrac{1}{4}\theta^3.$$

This algebraic number satisfies

$$4\psi^5 - 80\psi^4 + 100\psi^3 + 320\psi^2 - 355\psi + 4 = 0,$$

and $\mathfrak{q}$ is the denominator of the ideal $\langle\psi\rangle$. Hence the Thue equation (11) can be written as the norm form equation $4N_{\mathbb{L}/\mathbb{Q}}(X - Y\psi) = 1$, leading again to (13), where now $\langle\mu\rangle = \mathfrak{q}^{-1}$. In fact, we can take

$$\mu = -1 + \tfrac{1}{2}\theta.$$

Note that (12) is of the same form as (13), on putting $\mu = 1$, but of course with different parameters $\psi, \epsilon_1, \ldots, \epsilon_4$.

4. Upper bounds for linear forms in logarithms. In this section we will derive an absolute upper bound for

$$A = \max\{|a_1|, |a_2|, |a_3|, |a_4|\}.$$

We denote conjugates by upper indices in parentheses. Let $i, j, k \in \{1, 2, 3, 4, 5\}$ be pairwise distinct. Then the so-called Siegel identity reads

$$(\psi^{(j)} - \psi^{(k)})(X - Y\psi^{(i)}) + (\psi^{(k)} - \psi^{(i)})(X - Y\psi^{(j)}) + (\psi^{(i)} - \psi^{(j)})(X - Y\psi^{(k)}) = 0.$$

Let $i_0 \in \{1, 2, 3, 4, 5\}$ be such that

$$|X - Y\psi^{(i_0)}| = \min_{i \in \{1,2,3,4,5\}} |X - Y\psi^{(i)}|.$$

Note that a priori i_0 is unknown, since it depends on the unknown solution X, Y. Using (13) we can rewrite this Siegel identity to

$$\frac{\psi^{(i_0)} - \psi^{(j)}}{\psi^{(i_0)} - \psi^{(k)}} \frac{\mu^{(k)}}{\mu^{(j)}} \left(\frac{\epsilon_1^{(k)}}{\epsilon_1^{(j)}}\right)^{a_1} \left(\frac{\epsilon_2^{(k)}}{\epsilon_2^{(j)}}\right)^{a_2} \left(\frac{\epsilon_3^{(k)}}{\epsilon_3^{(j)}}\right)^{a_3} \left(\frac{\epsilon_4^{(k)}}{\epsilon_4^{(j)}}\right)^{a_4} - 1 = \frac{\psi^{(j)} - \psi^{(k)}}{\psi^{(k)} - \psi^{(i_0)}} \frac{\mu^{(i_0)}}{\mu^{(j)}} \frac{X - Y\psi^{(i_0)}}{X - Y\psi^{(j)}}, \tag{14}$$

in which the right hand side is small by the definition of i_0. We write the left hand side of (14) as $e^{\Lambda} - 1$, and thus we find that $|\Lambda|$ is small. In fact, following [**3**], we obtain, assuming $|Y| > Y_2'$, that

$$|\Lambda| < K_1 e^{-K_2 A}, \tag{15}$$

where Y_2', K_1, K_2 are computed from the parameters of [**3**], as follows (we computed with more significant digits than presented below; numbers are rounded in the proper direction).

	eq. (9)	eq. (10)	eq. (11)
$Y_0 =$	1	1	1
$C_1 <$	0·024631	0·105164	0·054535
$C_2 >$	0·945917	0·346227	0·506407
$Y_1 =$	1	1	1
$C_3 <$	13·602182	9·271511	18·176982
$Y_2^* =$	1	2	2
$\mu_- \geq$	1	0·045896	0·043882
$\mu_+ \leq$	1	1·540732	3·099078
$C_4 <$	30·023740	184·794950	474·554070
$C_5 <$	0·788330	0·633587	0·633587
$K_1 = C_6 <$	$1{\cdot}201067 \times 10^7$	$8{\cdot}435682 \times 10^{11}$	$6{\cdot}548406 \times 10^{13}$
$K_2 = 5/C_5 >$	6·342522	7·891577	7·891577
$Y_2' =$	2	5	7

The next step is to apply the theory of linear forms in logarithms of algebraic numbers. The best result today is that of Baker and Wüstholz [1]. In view of $\Lambda \neq 0$, under the condition $A \geq 3$, it implies

$$|\Lambda| > e^{-C_7 \log A}, \tag{16}$$

where in computing C_7 we take the parameters of [1] as follows: the number of terms in the linear forms is $n = 5$, the degree of the relevant field is $d = 20$, and we estimated the heights of the occurring algebraic numbers as follows. In our case it's easy to show that the height function h' used in [1] is just the absolute logarithmic Weil height h, defined by

$$h(\alpha) = \frac{1}{d} \log a_0 \prod_{i=1}^{d} \max\{1, |\alpha^{(i)}|\},$$

where a_0 is the leading coefficient of α. To avoid having to compute leading coefficients, we used $h(\alpha/\beta) \leq h(\alpha) + h(\beta)$ with algebraic integers α, β. Thus, using $C_7 = 18 . 6! . 5^6 . 640^7 . \log 200$ times the product of five heights, we found (we computed with more significant digits than presented below; numbers are rounded in the proper direction):

	eq. (9)	eq. (10)	eq. (11)
$h\left(\frac{\psi^{(i_0)} - \psi^{(j)}}{\psi^{(i_0)} - \psi^{(k)}} \frac{\mu^{(k)}}{\mu^{(j)}}\right) <$	4·218935	3·545317	4·771794
$h\left(\frac{\epsilon_1^{(k)}}{\epsilon_1^{(j)}}\right) <$	1·374322	2·215789	2·215789
$h\left(\frac{\epsilon_2^{(k)}}{\epsilon_2^{(j)}}\right) <$	2·399801	2·238231	2·238231
$h\left(\frac{\epsilon_3^{(k)}}{\epsilon_3^{(j)}}\right) <$	2·946885	2·461462	2·461462
$h\left(\frac{\epsilon_4^{(k)}}{\epsilon_4^{(j)}}\right) <$	2·830685	2·776732	2·776732
$C_7 <$	$5{\cdot}519275 \times 10^{30}$	$5{\cdot}714471 \times 10^{30}$	$7{\cdot}691352 \times 10^{30}$

Now an absolute upper bound for A, under the condition $|Y| > Y_2'$, follows at once from the inequalities (15) and (16). Summarizing, we find

$$\text{if } |Y| > Y_2' \quad \text{then} \quad A < K_3, \tag{17}$$

with Y_2', K_3 as in the following table.

	eq. (9)	eq. (10)	eq. (11)
$Y_2' =$	2	5	7
$K_3 <$	$6{\cdot}372677 \times 10^{31}$	$5{\cdot}289417 \times 10^{31}$	$7{\cdot}148610 \times 10^{31}$

5. Reduction of upper bounds. We take conjugates as follows: in the case of equation (9):

$$\theta^{(1)} = -3{\cdot}02\ldots, \theta^{(2)} = -2{\cdot}54\ldots, \theta^{(3)} = 0.67\ldots, \theta^{(4)} = 1{\cdot}44\ldots, \theta^{(5)} = 3{\cdot}44\ldots,$$

and in the case of equations (10) and (11)

$$\theta^{(1)} = -4{\cdot}19\ldots, \theta^{(2)} = -3{\cdot}69\ldots, \theta^{(3)} = 1{\cdot}10\ldots, \theta^{(4)} = 1{\cdot}91\ldots, \theta^{(5)} = 4{\cdot}88\ldots.$$

The method of [**3**] suggests that for each $i_0 \in \{1,2,3,4,5\}$ we pick arbitrary j, k, and work with the linear form Λ defined above. Following [**3**], for an inhomogeneous linear form in r unknowns (here $r = 4$) we have to compute the logarithms in this linear form to at least a precision of that of K_3^r (in our case about 125 decimal digits). Then the reduced upper bound will in general be proportional to $r \log K_3$ (a reasonable estimate in our example is about 40, we believe).

Yuri Bilu came up with the idea of using several independent linear forms simultaneously. Below we show how one can proceed in our example. In general (that is, in the totally real case; a similar idea should work in other cases as well) it is not difficult to show that one needs a precision of only about that of $K_3^{1+1/(r-1)}$ (only about 42 decimal digits in our situation), and one reaches a reduced upper bound proportional to $\frac{1}{r}\log K_3$ (below we find only 7 as the reduced upper bound in our example). In other words, we get more reduction for less money.

For each $i_0 \in \{1,2,3,4,5\}$ we take $j = i_0 + 1$ (modulo 5), and we consider the three linear forms Λ_1, Λ_2, Λ_3 corresponding to $k = i_0 + 2$, $i_0 + 3$, $i_0 + 4$ (modulo 5) respectively. Let us write the linear forms as

$$\Lambda_n = \alpha_{n,0} + \sum_{m=1}^{4} a_m \alpha_{n,m} \quad (n = 1,2,3).$$

These linear forms are independent in the sense that the matrix $(\alpha_{n,m})_{n=1,2,3,m=1,2,3,4}$ has rank 3. For each of the three forms Λ_n the inequality (15) is valid.

Take a convenient large enough number C, of about the size of $K_3^{4/3}$. In practice we took $C = 10^{48}$. Consider the lattice

$$\Gamma = \{\mathscr{A}\boldsymbol{x} \mid \boldsymbol{x} \in \mathbb{Z}^4\}$$

where

$$\mathscr{A} = \begin{pmatrix} [C\alpha_{1,1}] & [C\alpha_{1,2}] & [C\alpha_{1,3}] & [C\alpha_{1,4}] \\ [C\alpha_{2,1}] & [C\alpha_{2,2}] & [C\alpha_{2,3}] & [C\alpha_{2,4}] \\ [C\alpha_{3,1}] & [C\alpha_{3,2}] & [C\alpha_{3,3}] & [C\alpha_{3,4}] \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and consider the point

$$\boldsymbol{y} = \begin{pmatrix} -[C\alpha_{1,0}] \\ -[C\alpha_{2,0}] \\ -[C\alpha_{3,0}] \\ 0 \end{pmatrix}.$$

Here $[\cdot]$ denotes rounding to an integer (in practice we truncate towards zero).

For a quadruple $a_1, a_2, a_3, a_4 \in \mathbb{Z}$ we define λ_n for $n = 1, 2, 3$ by

$$\mathscr{A}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} - y = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ a_4 \end{pmatrix}.$$

Then we have

$$|\lambda_n - C\Lambda_n| \le |[C\alpha_{n,0}] - C\alpha_{n,0}| + \sum_{m=1}^{4} |a_m|\,|[C\alpha_{n,m}] - C\alpha_{n,m}|$$

$$\le 1 + 4A < 1 + 4K_3,$$

so assuming that the Λ_n are all near to 0, the length of this vector $(\lambda_1, \lambda_2, \lambda_3, a_4)^T$ is at most of the size of K_3. But in general the distance from a given lattice Γ to a given point y is of the size of $(\det\Gamma)^{1/\dim\Gamma}$, which is in our case of the size of $(C^3)^{1/4} \approx K_3$. So we might hope for a contradiction if C is large enough. This implies that at least one of the Λ_n is not near to 0, and in view of (15), that yields a reduced upper bound for A.

Computing a good lower bound for the distance $d(\Gamma, y)$ from Γ to y can be done by the LLL-algorithm, see [3] for details. Our computations led to the following lower bounds for $d(\Gamma, y)$.

i_0	eq. (9)	eq. (10)	eq. (11)
1	$2{\cdot}957854 \times 10^{34}$	$6{\cdot}318500 \times 10^{35}$	$5{\cdot}103141 \times 10^{35}$
2	$1{\cdot}383154 \times 10^{35}$	$1{\cdot}101822 \times 10^{35}$	$2{\cdot}690488 \times 10^{35}$
3	$3{\cdot}163028 \times 10^{34}$	$6{\cdot}373886 \times 10^{35}$	$2{\cdot}965987 \times 10^{35}$
4	$3{\cdot}266337 \times 10^{35}$	$2{\cdot}930310 \times 10^{35}$	$2{\cdot}963976 \times 10^{35}$
5	$2{\cdot}181709 \times 10^{35}$	$3{\cdot}016597 \times 10^{35}$	$2{\cdot}330763 \times 10^{35}$

It follows by

$$d(\Gamma, y)^2 \le \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + a_4^2 < 3\max\{\lambda_1^2, \lambda_2^2, \lambda_3^2\} + K_3^2,$$

that for some $n \in \{1, 2, 3\}$ we must have

$$|\lambda_n| > \sqrt{\tfrac{1}{3}(d(\Gamma, y)^2 - K_3^2)},$$

and hence

$$|\Lambda_n| > \frac{1}{C}\left(\sqrt{\tfrac{1}{3}(d(\Gamma, y)^2 - K_3^2)} - (1 + 4K_3)\right).$$

With (15) this yields a new upper bound for A. Upon substituting the bounds for K_1, K_2, K_3 and $d(\Gamma, y)$ given above, and $C = 10^{48}$, in all cases this new upper bound turned out to be only 7.

It remains to find the solutions with $A \le 7$ or $|Y| \le Y_2'$. This is straightforward, and produced only the solution $X = 1$, $Y = 0$, $a_1 = a_2 = a_4 = 0$ in the case of equation (9). This completes our proof.