Periodicity of p-adic continued fractions

A well known theorem of Lagrange states that the simple continued fraction expansion of a real number is periodic if and only if that real number is quadratic irrational. Several authors have tried to establish analogous results for continued fractions of p-adic numbers. The present author [3] showed that such a result is possible if one starts with sequences of approximation lattices of p-adic numbers, instead of continued fractions. We note that from a periodic sequence of approximation lattices of a p-adic number ξ it is easy to construct a periodic continued fraction expansion of ξ.

This process of constructing the continued fraction such that it is periodic, is the reverse of the process in the real case, where one starts by defining the continued fraction, and then tries to prove its periodicity. It would be interesting to obtain periodicity results for a given p-adic continued fraction expansion method, e.g. that introduced by Schneider [2]. Bundschuh [1] remarks that for this type of p-adic continued fractions a periodic continued fraction represents either a rational p-adic number of special type, or a quadratic irrational p-adic number (analogous to Euler's theorem). Further, he gives some numerical evidence indicating that the converse (analogous to Lagrange's theorem) may not be true.

It is the purpose of this note to show that for Schneider's continued fraction algorithm for p-adic numbers, it may indeed happen that quadratic irrational numbers in $\mathbb{Q}_p$ have non-periodic continued fraction expansions. Thus for this type of continued fractions an analogue of Lagrange's theorem is not true. Some criteria (but not exhaustive) will be given for (non)-periodicity. The examples that Bundschuh treated numerically are proved to be non-periodic.

Let p be prime. Let $\xi \in \mathbb{Q}_p$ be a nonzero p-adic integer. Schneider [2] defines the continued fraction expansion of ξ as follows. Put $\xi_0 = \xi$. Given a p-adic integer ξ_n for some $n \in \mathbb{N}_0$, let $a_n \in \{0, 1, \ldots, p-1\}$ be such that $b_n = \mathrm{ord}_p(\xi_n - a_n)$ is positive. We continue only if $\xi_n \neq a_n$. Then define ξ_{n+1} by

$$\xi_n = a_n + \frac{p^{b_n}}{\xi_{n+1}}.$$

Then $|\xi_n|_p = 1$ and $a_n \neq 0$ for all $n > 0$. Now the continued fraction expansion of ξ is

$$\xi = a_0 + \frac{p^{b_0}|}{|a_1} + \frac{p^{b_1}|}{|a_2} + \ldots.$$

Note that Bundschuh [1] has a slightly, but not essentially, different definition and notation.

We say that the continued fraction expansion of ξ is periodic if there exist $m_0 \in \mathbb{N}_0$, $k \in \mathbb{N}$ such that $a_{m+k} = a_m$, $b_{m+k} = b_m$ for all $m \geqq m_0$. Bundschuh [1] asks to prove or disprove periodicity for the continued fractions of $\xi = \sqrt{c}$, for $c \in \mathbb{Z}$ not a square, but a quadratic residue (mod p), so that $\xi \in \mathbb{Q}_p$, a.o. for the four examples $(c, p) = (-1, 5)$, $(2, 7)$, $(5, 11)$, $(3, 13)$.

There exist unique rational numbers P_n, Q_n such that

$$\xi_n = \frac{P_n + \sqrt{c}}{Q_n}$$

for $n \in \mathbb{N}_0$. Then $P_0 = 0$, $Q_0 = 1$, and we have the recursion formulas

$$\begin{aligned} P_{n+1} &= -(P_n - a_n \cdot Q_n), \\ Q_{n+1} &= (c - P_{n+1}^2)/p^{b_n} \cdot Q_n. \end{aligned}$$

We show that P_n and Q_n are integers, and that $Q_n \mid c - P_n^2$, for all $n \in \mathbb{N}_0$. It is obvious for $n = 0$. Suppose it is true for some $n \geqq 0$. Then P_{n+1} is obviously an integer. Further,

$$c - P_{n+1}^2 = c - P_n^2 + 2 \cdot a_n \cdot P_n \cdot Q_n - a_n^2 \cdot Q_n^2 \equiv 0 \pmod{Q_n},$$

and

$$\begin{aligned} c - P_{n+1}^2 &= (P_{n+1} + \sqrt{c}) \cdot (-P_{n+1} + \sqrt{c}) \\ &= (P_{n+1} + \sqrt{c}) \cdot (P_n - a_n Q_n + \sqrt{c}) \\ &= (P_{n+1} + \sqrt{c}) \cdot (\xi_n - a_n) \cdot Q_n \equiv 0 \pmod{p^{b_n}}, \end{aligned}$$

hence Q_{n+1} is integral. It follows at once that $Q_{n+1} \mid c - P_{n+1}^2$. We now prove the following lemma.

Lemma. *If for some n the signs of P_n and Q_n are different, and $P_{n+1}^2 > c$, then the continued fraction of $\sqrt{c} \in \mathbb{Q}_p$ is non-periodic.*

Proof. Since P_n and Q_n have different signs, it follows by the recursion formula for P_{n+1} that P_n and P_{n+1} have different signs. By the recursion formula for Q_{n+1} and by $P_{n+1}^2 > c$ it follows that $Q_{n+1} \neq 0$, and Q_n and Q_{n+1} have different signs. Hence the signs of P_{n+1} and Q_{n+1} are different. Further, by $a_{n+1} \neq 0$,

$$|P_{n+2}| = |P_{n+1}| + a_{n+1} \cdot |Q_{n+1}| > |P_{n+1}|,$$

so that $P_{n+2}^2 > c$. Consequently, the conditions for the lemma hold for $n+1$ as well. Hence, by induction, $|P_n| \to \infty$ as $n \to \infty$. But periodicity of the continued fraction implies periodicity of P_n and Q_n, which contradicts that $|P_n|$ is unbounded. □

Corollary. *If $c < 0$ then the continued fraction of $\sqrt{c} \in \mathbb{Q}_p$ is non-periodic.*

Proof. If $c < 0$ then always $P_n^2 > c$. Further, $P_1 = a_0 > 0$, and $Q_1 = (c - P_1^2)/p^{b_0} < 0$. Apply the lemma for $n = 1$. □

Examples. The four examples of Bundschuh all satisfy the conditions for the above lemma with $n = 1$. We give a few details below.

1. $\sqrt{-1} \in \mathbb{Q}_5$. The non-periodicity follows from the corollary. We have in fact, if we take $\sqrt{-1} \equiv 2 \pmod 5$,

n	P_n	Q_n	a_n	b_n
0	0	1	2	1
1	2	−1	1	1
2	−3	2	2	2

and the continued fraction starts with

$$2 + \frac{5|}{|1} + \frac{5|}{|2} + \frac{25|}{|1} + \frac{5|}{|3} + \frac{5|}{|4} + \frac{5|}{|2} + \frac{5|}{|2} + \frac{5|}{|1} + \frac{125|}{|1} + \frac{5|}{|4} + \dots .$$

2. $\sqrt{2} \in \mathbb{Q}_7$. We take $\sqrt{2} \equiv 3 \pmod 7$. Then we have

n	P_n	Q_n	a_n	b_n
0	0	1	3	1
1	3	−1	1	1
2	−4	2	3	2

3. $\sqrt{5} \in \mathbb{Q}_{11}$. We take $\sqrt{5} \equiv 4 \pmod{11}$. Then we have

n	P_n	Q_n	a_n	b_n
0	0	1	4	1
1	4	-1	3	1
2	-7	4	2	1

4. $\sqrt{3} \in \mathbb{Q}_{13}$. We take $\sqrt{3} \equiv 4 \pmod{13}$. Then we have

n	P_n	Q_n	a_n	b_n
0	0	1	4	1
1	4	-1	5	1
2	-9	6	10	1

Next we show that there are also many p-adic quadratic irrationals that do have periodic continued fraction expansions, as defined above. Let $c \in \mathbb{N}$ be a non-square, that can be written as $c = e^2 + d \cdot p^k$ for $d, e, k \in \mathbb{N}$ with $1 \leqq e \leqq \frac{1}{2}(p-1)$ and $d \mid 2e$, $p \nmid d$. Then we find

n	P_n	Q_n	a_n	b_n
0	0	1	e	k
1	e	d	$2e/d$	k
2	e	1	$2e$	k
3	e	d	$2e/d$	k

so that the continued fraction is periodic with period length 2.
We conclude by giving some «exceptional» pairs (c, p) which do have periodic continued fractions, but seem not to fit in an infinite sequence, such as given above. They are: $(c, p) = (136, 3), (376, 5), (148, 7), (388, 11)$. We have (with the bar denoting the repeating part):

$$\sqrt{136} = 1 + \overline{\frac{27|}{|1} + \frac{3|}{|1} + \frac{3|}{|1} + \frac{27|}{|2}},$$

$$\sqrt{376} = 1 + \overline{\frac{125|}{|4} + \frac{5|}{|1} + \frac{5|}{|3} + \frac{5|}{|1} + \frac{5|}{|4} + \frac{125|}{|2}},$$

$$\sqrt{148} = 1 + \overline{\frac{49|}{|3} + \frac{7|}{|4} + \frac{7|}{|3} + \frac{49|}{|2}},$$

$$\sqrt{388} = 5 + \overline{\frac{121|}{|7} + \frac{11|}{|8} + \frac{11|}{|7} + \frac{121|}{|10}}.$$

Suggestions for further research. We observe that in all cases presented above, as in the real case, the periodic continued fractions are symmetric. Namely, for period length k we have $a_j = a_{k-j}$ and $b_j = b_{k+1-j}$ for $j = 1, 2, \dots, k-1$. Further, $a_k = 2 \cdot a_0$ appears to hold, as in the real case. We found no examples of odd period length.

There is also a connection to Pell-like equations. We illustrate this with the example $\sqrt{376} \in \mathbb{Q}_5$. Let p_n/q_n be the nth convergent of the continued fraction expansion, defined by

$$p_{-1} = 1, \; p_0 = a_0, \; p_n = a_n \cdot p_{n-1} + 5^{b_n} \cdot p_{n-2} \quad \text{for } n \geqq 1,$$
$$q_{-1} = 0, \; q_0 = 1, \; q_n = a_n \cdot q_{n-1} + 5^{b_n} \cdot q_{n-2} \quad \text{for } n \geqq 1.$$

Put

$$p_n^2 - 376 \cdot q_n^2 = d_n \cdot 5^{c_n}, \; c_n \in \mathbb{N}, \; d_n \in \mathbb{Z}, \; 5 \nmid d_n.$$

Then we find that $c_n = \sum_{j=0}^{n} b_j$, and the sequence $\{d_n\}_{n=-1}^{\infty}$ is given by 1, -3, 17, -4, 17, -3, 1, -3, 17, ..., which is symmetric. The fifth convergent $p_5/q_5 = 12\,103/603$, for which $d_5 = 1$, gives rise to a sort of 5-adic ‹fundamental unit› $12\,103 + 603 \cdot \sqrt{376}$, in the sense that $p_{i+6j} + q_{i+6j} \cdot \sqrt{376} = (p_i + q_i \cdot \sqrt{376}) \cdot (p_5 + q_5 \cdot \sqrt{376})^j$ for $i = -1, 0, \dots, 4$, and $j = 0, 1, 2, \dots$. It would be interesting to have a more general theory of these matters.

B. M. M. de Weger, Faculty of Applied Mathematics,
University of Twente, Enschede

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