Chapter 5. The inequality $0 < x - y < y^{\delta}$ in S-integers.

The results of this chapter have been published in de Weger [1987[a]].

5.1. Introduction.

Let S be the set of all positive integers composed of primes from a fixed finite set $\{ p_1, \dots, p_s \}$, where $s \ge 2$, and let $\delta \in (0,1)$. In this chapter we study the diophantine inequality

$$0 < x - y < y^{\delta} \tag{5.1}$$

in $x, y \in S$. We give explicit upper bounds for the solutions, and we show how the algorithms for homogeneous, one- and multi-dimensional diophantine approximation in the real case, that were presented in Chapter 3, can be used for finding all solutions of (5.1) for any set of parameters $p_1, \dots, p_s$, δ. For $s = 2$ the continued fraction method (cf. Section 3.2) is used. For $s \ge 3$ we use the L^3-algorithm for reducing upper bounds (cf. Section 3.7).

Tijdeman [1973] (see also Shorey and Tijdeman [1986], Theorem 1.1) showed that there exists a computable number c, depending on $\max(p_i)$ only, such that for all $x, y \in S$ with $x > y \ge 3$,

$$x - y > y/(\log y)^c .$$

Thus, for any solution of (5.1) a bound for x, y follows. Størmer [1897] showed how to solve the equation $x - y = k$ with $k = 1, 2$ with an elementary method (see also Mahler [1935], Lehmer [1964]). Our method can solve this equation for arbitrary $k \in \mathbb{Z}$. For the one-dimensional case $s = 2$, Ellison [1971[b]] has proved the following result: for all but finitely many explicitly given exceptions, $| 2^x - 3^y | > \exp(x \cdot (\log 2 - 1/10))$ for all $x, y \in \mathbb{N}$. Cijsouw, Korlaar and Tijdeman (appendix to Stroeker and Tijdeman [1982]) have found all the solutions $x, y \in \mathbb{N}$ of the inequality

$$| p^x - q^y | < p^{\delta \cdot x}$$

for all primes p, q with $p < q < 20$, and with $\delta = \frac{1}{2}$. We shall extend these results for many more values of p, q and with $\delta = 0.9$. Further, we determine all the solutions of (5.1) for the multi-dimensional case $t = 6$, $\{ p_1, \dots, p_6 \} = \{ 2, 3, 5, 7, 11, 13 \}$ with $\delta = \frac{1}{2}$.

In Section 5.2 we derive upper bounds for the solutions of (5.1). In Sections 5.3 and 5.4 we give a method for reducing such upper bounds in the one- and multi-dimensional cases respectively, and work them out explicitly for some examples. Section 5.5 contains tables with numerical data.

5.2. Upper bounds for the solutions.

We assume that the primes are ordered as $p_1 < \dots < p_s$. For a solution x, y of (5.1), the finitely many $z \in \mathbb{N}$ for which $z \cdot x$, $z \cdot y$ is also a solution of (5.1) can be found without any difficulty. Therefore we may assume that $(x,y) = 1$. Put

$$X = \max_{1 \le i \le s} \mathrm{ord}_{p_i}(x \cdot y) .$$

Put

$$C_1 = 2^{9 \cdot s+26} \cdot s^{s+4} \cdot \max(1, \frac{1}{\log p_1}) \cdot \Big(\prod_{i=2}^{s} \log p_i \Big) \cdot \log(e \cdot \log p_{s-1})/(1-\delta) ,$$

$$C_2 = 2 \cdot \log 2/\log p_1 + 2 \cdot C_1 \cdot \log(e \cdot C_1 \cdot \log p_s) .$$

<u>THEOREM 5.1.</u> *The solutions of* (5.1) *satisfy* $X < C_2$.

<u>Proof.</u> If $y \le \frac{1}{2} \cdot x$, then $y^{\delta} > x - y \ge y$, which contradicts $y \ge 1$. So $y > \frac{1}{2} \cdot x$. Put $\Lambda = \log(x/y)$. Then

$$0 < \Lambda < x/y - 1 < y^{-(1-\delta)} < (\tfrac{1}{2} \cdot x)^{-(1-\delta)} . \tag{5.2}$$

By $x = \max(x,y) \ge p_1^X$, we obtain

$$0 < \Lambda < 2^{1-\delta} \cdot p_1^{-(1-\delta) \cdot X} . \tag{5.3}$$

We apply Waldschmidt's result, Lemma 2.4, to Λ , with $n = s$, $q = 2$. Note that the 'independence condition' $[\mathbb{Q}(\sqrt{p_1}, \dots, \sqrt{p_n}) : \mathbb{Q}] = 2^n$ holds. Since $p_i \ge 3$ we have $V_i = \log p_i$ for $i \ge 2$. Thus

$$\Lambda > \exp\left(-(\log X + \log(e\cdot\log p_s))\cdot C_1\cdot(1-\delta)\cdot\log p_1 \right) .$$

Combining this with (5.3) we find

$$X < C_1\cdot\log(e\cdot\log p_s) + \log 2/\log p_1 + C_1\cdot\log X .$$

The result now follows from Lemma 2.1, since $C_1 > e^2$. □

Examples. With $s = 2$, $2 \le p_i \le 199$, $\delta = 0.9$ we have $C_1 < 2.30\times10^{17}$, $C_2 < 1.97\times10^{19}$.
With $s = 6$, $2 \le p_i \le 13$, $\delta = \frac{1}{2}$ we find $C_1 < 8.37\times10^{33}$, $C_2 < 1.35\times10^{36}$.

5.3. Reducing the upper bounds in the one-dimensional case.

In this section we work out the examples $s = 2$, $\delta = 0.9$, and p_1, p_2 run through either the set of primes below 200, or the set of non-powers below 50. We note that for any other set of parameters p_1, p_2, δ the method works similarly. We prove the following result.

THEOREM 5.2. (a) *The diophantine inequality*

$$| p_1^{x_1} - p_2^{x_2} | < \min\left(p_1^{x_1}, p_2^{x_2} \right)^{\delta} \tag{5.4}$$

with p_1, p_2 *primes such that* $p_1 < p_2 < 200$, *and*

$$\begin{aligned} &x_1, x_2 \in \mathbb{Z},\ x_1 \ge 2,\ x_2 \ge 2,\ \textit{and either}\ \delta = \tfrac{1}{2} \\ &\quad \textit{or}\ \delta = 0.9,\ \min\left(p_1^{x_1}, p_2^{x_2} \right) > 10^{15} \end{aligned} \tag{5.5}$$

has only the 77 solutions listed in Table I.
(b) *The diophantine inequality* (5.4) *with* p_1, p_2 *non-powers such that* $2 \le p_1 < p_2 \le 50$ *and conditions* (5.5), *has only the 74 solutions listed in* Table II.

Remarks. The Tables are given in Section 5.5. In Tables I, II the column "delta" gives the real number with

$$| p_1^{x_1} - p_2^{x_2} | = \min\left(p_1^{x_1}, p_2^{x_2} \right)^{\text{delta}} .$$

Note that in Theorem 5.2 we do not demand $(x_1,x_2) = 1$, and in Theorem

5.2(b) we do not demand p_1, p_2 to be primes. The conditions (5.5) are chosen, since the numerous solutions of (5.4) with $\delta = 0.9$ and $\min\left(p_1^{x_1}, p_2^{x_2} \right) \le 10^{15}$ can be found without much effort.

<u>Proof.</u> Write

$$\Lambda = | x_1 \cdot \log p_1 - x_2 \cdot \log p_2 | , \quad X = \max(x_1,x_2) .$$

We assume that

$$p_1^X > 10^{25} , \tag{5.6}$$

since it is easy to find the remaining solutions. Let $\log p_1/\log p_2$ have the simple continued fraction expansion (cf. Section 3.2)

$$\log p_1/\log p_2 = [\, 0, a_1, a_2, \ldots \,] ,$$

and let the convergents be r_n/q_n for $n = 1, 2, \ldots$. We may assume that $(x_1,x_2) = 1$. First we show that $x_1 \ge x_2$, hence $X = x_1$. For if $x_1 < x_2$, then

$$\Lambda = x_2 \cdot \log p_2 - x_1 \cdot \log p_1 > X \cdot \left(\log p_2 - \log p_1 \right) \ge X \cdot \log \frac{199}{197} ,$$

and from (5.3) and (5.6) we then infer

$$0.0101 \le 0.0101 \cdot X < X \cdot \log \frac{199}{197} < \Lambda < 2^{0.1} \cdot 10^{-5/2} < 0.0034 ,$$

which is contradictory. Next we prove that

$$p_1^{X/10} > 3.1 \cdot X . \tag{5.7}$$

Namely, suppose the contrary. Then $2^{X/10} \le 3.1 \cdot X$, and it follows that $X \le 80$. This contradicts $3.1 \cdot X \ge p_1^{X/10} > 10^{5/2}$. From (5.3) we infer

$$\left| \frac{x_2}{X} - \frac{\log p_1}{\log p_2} \right| < \frac{2^{0.1}}{\log p_2} \cdot p_1^{-X/10} \cdot \frac{1}{X} . \tag{5.8}$$

It follows from (5.7) that

$$\left| \frac{x_2}{X} - \frac{\log p_1}{\log p_2} \right| < \frac{2^{0.1}}{\log 2} \cdot \frac{1}{3.1 \cdot X^2} < \frac{1}{2 \cdot X^2} .$$

Hence x_2/X is, by Lemma 3.1, a convergent of $\log p_1/\log p_2$, say r_k/q_k . From the example at the end of Section 5.2 we see that $X \le X_0 < 1.97 \times 10^{19}$.

We find from (3.7) that $k \le 92.996$, hence $k \le 92$. Lemma 3.1 further yields: if (5.3) holds then

$$a_{k+1} > -2 + p_1^{q_k/10} \cdot \frac{1}{q_k} \cdot \frac{\log p_2}{2^{0.1}} , \tag{5.9}$$

and if

$$a_{k+1} > p_1^{q_k/10} \cdot \frac{1}{q_k} \cdot \frac{\log p_2}{2^{0.1}} \tag{5.10}$$

then (5.3) holds for $(x_1, x_2) = (q_k, r_k)$. We computed the continued fraction expansions and the convergents of all numbers $\log p_1/\log p_2$ in the mentioned ranges for p_1, p_2 exactly up to the index n such that $q_{n-1} \le 1.97 \times 10^{19} < q_n$ (cf. Section 2.5 for details of the computational method). Note that $n \le 93$. We checked all convergents for (5.9), and subsequently for (5.10). It is possible, though unlikely, that there is a convergent that satisfies (5.9) but fails (5.10). We met only one such a case: $p_1 = 15$, $p_2 = 23$, with $\log 15/\log 23 = [\ 0,\ 1,\ 6,\ 2,\ 1,\ 51,\ \ldots]$, so that $a_5 = 51$, $r_4 = 19$, $q_4 = 22$. Now, (5.9) holds but (5.10) fails, since

$$15^{2.2} \cdot \frac{1}{22} \cdot (\log 19)/2^{0.1} = 51.4\ldots \in [51, 53) .$$

We have in this case $\Lambda = 0.002714\ldots < 0.002771\ldots = 2^{0.1} \cdot 15^{-2.2}$, so (5.3) is true. But $\log(15^{22} - 23^{19})/\log(23^{19}) = 0.9008\ldots > \delta$, so (5.1) is not true. This example illustrates that (5.3) is weaker than (5.1). Therefore all found solutions of (5.3) have been checked for (5.1) as well. The proof is now completed by the details of the computations, which we do not give here. □

The computations for the proof of Theorem 5.2 took 35 sec.

5.4. Reducing the upper bounds in the multi-dimensional case.

Now let $s \ge 3$. Put $x_i = \mathrm{ord}_{p_i}(x/y)$ for $i = 1, \ldots, s$. Then $X = \max|x_i|$, and

$$\Lambda = \sum_{i=1}^{s} x_i \cdot \log p_i .$$

Note that (5.3) is of the form (3.1). Hence by Theorem 5.1 we can use the method described in Section 3.7 for solving (5.3). We shall do so for the example $t = 6$, $\{ p_1, \ldots, p_6 \} = \{ 2, 3, 5, 7, 11, 13 \}$ (the first six primes), and $\delta = \frac{1}{2}$.

We use small refinements of Lemmas 3.7 and 3.8, devised specially for this application, as follows. Let notation be as in Section 3.7.

LEMMA 5.3. *Let* X_1 *be a positive number such that*

$$\ell(\Gamma) \geq \sqrt{\left(4 \cdot n^2 + (n-1) \cdot \gamma^2\right)} \cdot X_1 \ . \tag{5.11}$$

Then (5.3) *has no solutions with for* $i = 1, \ldots, s$

$$\log\left(\gamma \cdot C \cdot \sqrt{2}/s \cdot X_1\right) / \tfrac{1}{2} \cdot \log p_i \leq |x_i| \leq X \leq X_1 \ . \tag{5.12}$$

LEMMA 5.4. *Suppose that*

$$|\tilde{\Lambda}| > \sum_{i=1}^{s} |x_i| \ . \tag{5.13}$$

Then

$$|x_i| < \log\left[\gamma \cdot C \cdot \sqrt{2}/\left(|\lambda| - \sum_{i=1}^{s} |x_i|\right)\right] / (1-\delta) \cdot \log p_i \ . \tag{5.14}$$

Remark. Lemmas 5.3 and 5.4 are refinements of Lemma 3.8, in that they differentiate between the different x_i . Moreover, Lemma 5.3 has slightly sharper condition and conclusion than Lemma 3.7.

Proofs (of Lemmas 5.3 and 5.4). Analogous to the proofs of Lemmas 3.7 and 3.8, using (5.2) and

$$p_i^{|x_i|} \leq \max(x,y) = x < 2 \cdot |\Lambda|^{-1/2} \ . \qquad \square$$

THEOREM 5.5. *The diophantine inequality*

$$0 < x - y < \sqrt{y}$$

in $x, y \in S = \{ 2^{x_1} \cdot \ldots \cdot 13^{x_6} \mid x_i \in \mathbb{N}_0 \text{ for } i = 1, \ldots, 6 \}$ *with* $(x,y) = 1$ *has exactly* 605 *solutions. Among those,* 571 *satisfy*

$$\text{ord}_2(x \cdot y) \leq 19 \ , \quad \text{ord}_3(x \cdot y) \leq 12 \ , \quad \text{ord}_5(x \cdot y) \leq 8 \ ,$$

$$\mathrm{ord}_7(x\cdot y) \le 7\ ,\quad \mathrm{ord}_{11}(x\cdot y) \le 5\ ,\quad \mathrm{ord}_{13}(x\cdot y) \le 5\ .$$

The remaining 34 *solutions are listed in* Table III.

Remark. The upper bounds for $\mathrm{ord}_{p_i}(xy)$ given for the 571 solutions not listed in Table III are chosen such that it takes a reasonable amount of computer time to find them all by a brute force method. The list of all 605 solutions is too extensive to be reproduced here.

Proof. By the example at the end of Section 5.2 we know that $X < X_0$ for $X_0 = 1.35\times10^{36}$. We apply the method described in Section 3.7. Take $C = 10^{240}$ (which is chosen so that it is somewhat larger than X_0^6), and $\gamma = 1$. We applied the L^3-algorithm to the corresponding lattice Γ_1, and found a reduced basis $\underline{c}_1, \ldots, \underline{c}_6$ with $|\underline{c}_1| > 9.40\times10^{39}$. So Lemma 3.4 yields

$$\ell(\Gamma_1) > 2^{-5/2}\cdot 9.40\times10^{39} > 1.66\times10^{39}\ .$$

This is larger than $\sqrt{(4\cdot 6^2+5\cdot 1^2)}\cdot X_0 = 1.64\ldots\times10^{37}$, so (5.11) holds with $X_1 = X_0$. By Lemma 5.3 we find

$$X < \log\left(10^{240}\cdot\sqrt{2}/6\cdot 1.35\times10^{36}\right)/\tfrac{1}{2}\cdot\log 2 < 1350.4\ ,$$

so $X \le 1350$. Next we choose $C = 10^{32}$, $\gamma = 1$, and $X_0 = 1350$. The reduced basis of the corresponding lattice Γ_2 was computed, and we found $|\underline{c}_1| > 2.71\times10^5$. Hence $\ell(\Gamma_2) > 4.79\times10^4$, which is larger than $\sqrt{149}\cdot 1350 = 1.64\ldots\times10^4$. Hence Lemma 5.3 yields for all $i = 1, \ldots, 6$

$$|x_i| < \log\left(10^{32}\cdot\sqrt{2}/6\cdot 1350\right)/\tfrac{1}{2}\cdot\log p_i\ ,$$

and it follows that

$$|x_1| \le 187\ ,\quad |x_2| \le 118\ ,\quad |x_3| \le 80\ ,$$
$$|x_4| \le 66\ ,\quad |x_5| \le 54\ ,\quad |x_6| \le 50\ . \tag{5.15}$$

Next we choose $C = 10^{12}$, $\gamma = 10^4$. We use Lemma 5.4 as follows. If $|\lambda| > 10^6$ then (5.13) holds by (5.15), and Lemma 5.4 yields

$$|x_1| \le 67\ ,\quad |x_2| \le 42\ ,\quad |x_3| \le 29\ ,$$
$$|x_4| \le 24\ ,\quad |x_5| \le 19\ ,\quad |x_6| \le 18\ . \tag{5.16}$$

All vectors in the corresponding lattice Γ_3 satisfying (5.15) and $|\lambda| < 10^6$ have been computed with the Fincke and Pohst algorithm, cf. Section 3.6. We omit details. We found that there exist only two such vectors, but they do not correspond to solutions of (5.1). Hence all solutions of (5.1) satisfy (5.16). Next, we choose $C = 10^8$, $\gamma = 10^4$. If $|\lambda| > 5\times10^5$ then Lemma 5.4 yields

$$|x_1| \le 42 , \quad |x_2| \le 27 , \quad |x_3| \le 18 ,$$
$$|x_4| \le 15 , \quad |x_5| \le 12 , \quad |x_6| \le 11 . \tag{5.17}$$

There are 143 vectors in the corresponding lattice Γ_4 satisfying (5.16) and $|\lambda| \le 5\times10^5$. Of them, 2 correspond to solutions of (4.1), namely those with

$$(x_1,\dots,x_6) = (\ 7, -5, \ 3, -9, -3, \ 8) , \quad \lambda = 257674 ,$$
$$(x_1,\dots,x_6) = (-10, 10, -6, \ 5, -6, \ 4) , \quad \lambda = 144817 .$$

Both also satisfy (5.17). Hence all solutions of (5.1) satisfy (5.17). At this point it seems inefficient to choose appropriate parameters C, γ, and a bound for $|\lambda|$ to repeat the procedure with. But the bounds of (5.17) are small enough to admit enumeration. Doing so, we found the result. □

<u>Remark.</u> Theorems 5.2 and 5.5 find applications in solving other exponential diophantine equations, see Stroeker and Tijdeman [1982], Alex [1985[a]], [1985[b]], Tijdeman and Wang [1987], and Section 6.4 of this thesis.

<u>Remark.</u> The computation of the reduced basis of Γ_1 took 113 sec, where we applied the L^3-algorithm as we described it in Section 3.5, in 12 steps. The direct search for the solutions of (5.17) took 228 sec. The remaining computations (computation of the $\log p_i$ up to 250 decimal digits, of the reduced basis of Γ_2, and of the short vectors in Γ_3 and Γ_4) took 8 sec. Hence in total we used 349 sec.

5.5. Tables.

Table I. (Theorem 5.2(a)) : see p. 114-115.

Table II. (Theorem 5.2(b)) : see p. 116-117.

Table III. (Theorem 5.5).

x_1	x_2	x_3	x_4	x_5	x_6	x	y	$x-y$
−1	11	−1	0	6	0	17 71561	17 71470	91
0	4	5	1	−6	0	17 71875	17 71561	314
21	−2	−2	−1	−3	0	20 97152	20 96325	827
1	13	−1	−3	−1	−2	31 88646	31 88185	461
19	0	0	−8	1	0	57 67168	57 64801	2367
6	2	−1	1	6	3	88 58304	88 57805	499
−2	15	−1	−2	−4	0	143 48907	143 48180	727
11	−15	0	2	1	1	143 50336	143 48907	1429
1	8	−1	−8	0	3	288 29034	288 24005	5029
−22	5	1	−1	1	3	293 62905	293 60128	2777
13	1	3	−1	1	−6	337 92000	337 87663	4337
1	2	9	−4	−4	0	351 56250	351 53041	3209
3	3	0	4	2	7	627 52536	627 48517	4019
−26	1	0	5	3	0	671 10351	671 08864	1487
3	−13	10	−2	0	0	781 25000	781 21827	3173
8	−2	−10	4	1	1	878 95808	878 90625	5183
25	1	−4	0	5	0	1006 63296	1006 56875	6421
−6	1	−2	−6	0	7	1882 45551	1882 38400	7151
8	−13	0	3	−2	3	1929 14176	1929 13083	1093
1	−13	−3	7	2	0	1992 97406	1992 90375	7031
−4	−1	−4	1	−4	7	4392 39619	4392 30000	9619
−4	2	−11	2	6	0	7812 58401	7812 50000	8401
16	−3	5	1	−1	−6	14336 00000	14335 62273	37727
−8	8	0	−8	3	2	14758 24779	14757 89056	35723
−5	−2	−5	11	0	−3	19773 26743	19773 00000	26743
−25	7	1	0	−2	5	40600 88955	40600 86272	2683
2	0	13	−9	−2	0	48828 12500	48827 86447	26053
−14	19	−2	−4	1	−1	1 27848 76137	1 27848 44800	31337
−24	−1	−2	12	−1	0	1 38412 87201	1 38412 03200	84001
−5	5	10	0	1	−8	2 61035 15625	2 61033 83072	1 32553
2	−4	−9	3	7	−2	2 67363 98612	2 67363 28125	70487
18	7	0	−13	0	2	9 68892 08832	9 68890 10407	1 98425
7	−5	3	−9	−3	8	1305 16915 36000	1305 16881 72831	33 63169
−10	10	−6	5	−6	4	2834 49801 04623	2834 49760 00000	41 04623

Table I. (Theorem 5.2(a)).

p_1	x_1	$p_1^{x_1}$	p_2	x_2	$p_2^{x_2}$	delta
2	3	8	3	2	9	0.00000
3	3	27	5	2	25	0.21534
2	5	32	3	3	27	0.48832
5	3	125	11	2	121	0.28906
2	7	128	11	2	121	0.40575
2	7	128	5	3	125	0.22754
2	8	256	3	5	243	0.46694
7	3	343	19	2	361	0.49512
2	9	512	23	2	529	0.45416
3	7	2187	13	3	2197	0.29941
3	7	2187	47	2	2209	0.40194
13	3	2197	47	2	2209	0.32293
19	3	6859	83	2	6889	0.38504
31	3	29791	173	2	29929	0.47828
2	15	32768	181	2	32761	0.18716
13	7	627 48517	89	4	627 42241	0.48703
2	50	1 12589 99068 42624	47	9	1 11913 04731 02767	0.85259
7	18	1 62841 35979 10449	149	7	1 63043 64614 03549	0.80898
19	12	2 21331 49190 66161	83	8	2 25229 22321 39041	0.88568
2	51	2 25179 98136 85248	19	12	2 21331 49190 66161	0.88532
2	51	2 25179 98136 85248	83	8	2 25229 22321 39041	0.76159
5	22	2 38418 57910 15625	157	7	2 35124 32775 37493	0.87942
13	14	3 93737 63856 99289	89	8	3 93658 88057 02081	0.76282
17	13	9 90457 80329 05937	193	7	9 97473 03260 05057	0.86560
7	19	11 39889 51853 73143	197	7	11 51499 04768 98413	0.87594
61	9	11 69414 60928 34141	197	7	11 51499 04768 98413	0.88743
5	23	11 92092 89550 78125	61	9	11 69414 60928 34141	0.89343
5	23	11 92092 89550 78125	29	11	12 20050 97657 05829	0.89862
29	11	12 20050 97657 05829	199	7	12 35866 42791 61399	0.88268
23	12	21 91462 44320 20321	43	10	21 61148 23132 84249	0.88656
11	16	45 94972 98635 72161	71	9	45 84850 07184 49031	0.84059
5	24	59 60464 47753 90625	73	9	58 87158 67082 67913	0.88642
37	11	177 91762 17794 60413	53	10	174 88747 03655 13049	0.89785
29	12	353 81478 32054 69041	89	9	350 35640 37074 85209	0.88568
23	13	504 03636 19364 67383	163	8	498 31141 43181 21121	0.89040
23	13	504 03636 19364 67383	59	10	511 11675 33006 41401	0.89536
11	17	505 44702 84992 93771	163	8	498 31141 43181 21121	0.89580

11	17	505 44702 84992 93771	23	13	504 03636 19364 67383	0.85578
11	17	505 44702 84992 93771	59	10	511 11675 33006 41401	0.88985
7	21	558 54586 40832 84007	41	11	550 32903 17162 48441	0.89708
19	14	799 00668 57828 84121	31	12	787 66278 37885 49761	0.89710
19	14	799 00668 57828 84121	173	8	802 35917 84760 91681	0.86722
2	60	1152 92150 46068 46976	181	8	1151 93665 78235 00641	0.83013
67	10	1822 83780 45517 61449	107	9	1838 45921 24201 54507	0.88680
47	11	2472 15921 50840 12303	199	8	2459 37419 15531 18401	0.87580
13	17	8650 41591 93813 37933	127	9	8594 75474 86093 97887	0.88441
2	63	9223 37203 68547 75808	53	11	9269 03592 93721 91597	0.87844
3	41	36472 99637 71707 86403	149	9	36197 31987 96201 91349	0.89170
2	65	36893 48814 74191 03232	5	28	37252 90298 46191 40625	0.89721
2	66	73786 97629 48382 06464	97	10	73742 41268 94928 26049	0.83799
3	42	1 09418 98913 15123 59209	101	10	1 10462 21254 11204 51001	0.89916
2	68	2 95147 90517 93528 25856	29	14	2 97558 23267 57994 63481	0.89800
113	10	3 39456 73899 22223 14849	191	9	3 38298 68155 95733 17311	0.87990
53	12	4 91258 90425 67261 54641	199	9	4 89415 46411 90705 61799	0.88284
5	30	9 31322 57461 54785 15625	41	13	9 25103 10231 50136 29321	0.89638
19	17	54 80386 85778 48021 85939	47	13	54 60999 70612 05831 77327	0.88730
23	16	61 32610 41568 09986 48961	151	10	61 62677 95033 67185 14001	0.89400
2	73	94 44732 96573 92904 27392	7	26	93 87480 33764 77543 05649	0.89920
2	75	377 78931 86295 71617 09568	181	10	377 38596 84695 57044 99801	0.86840
2	75	377 78931 86295 71617 09568	41	14	379 29227 19491 55588 02161	0.89368
41	14	379 29227 19491 55588 02161	181	10	377 38596 84695 57044 99801	0.89828
3	49	2392 99329 23061 75295 90083	17	19	2390 72435 68515 13248 47153	0.87071
13	21	2470 64529 07345 03927 04413	89	12	2469 90403 56526 21403 03521	0.84941
103	12	14257 60886 84617 89454 47841	157	11	14285 52404 46318 60195 25093	0.88788
3	51	21536 93963 07555 77663 10747	163	11	21580 60662 62396 00904 07387	0.88933
7	29	32199 05755 81317 97268 37607	13	22	32118 38877 95485 51051 57369	0.89390
11	24	98497 32675 80761 10947 11841	61	14	98768 32533 36131 80951 12441	0.89755
37	16	1 23375 11914 21716 63622 74241	191	11	1 23414 74201 97479 41888 22591	0.86078
2	84	1 93428 13113 83406 67952 98816	199	11	1 93813 41794 57931 33178 02199	0.89319
2	84	1 93428 13113 83406 67952 98816	3	53	1 93832 45667 68001 98967 96723	0.89402
3	53	1 93832 45667 68001 98967 96723	199	11	1 93813 41794 57931 33178 02199	0.84151
7	30	2 25393 40290 69225 80878 63249	31	17	2 25501 16774 16274 31786 82911	0.86903
2	90	123 79400 39285 38027 48991 24224	181	12	123 63541 71303 11583 51179 80561	0.89326
43	17	587 44031 06360 42001 88795 53643	71	15	587 32059 59385 49335 38673 30551	0.86709
2	99	63382 53001 14114 70074 83516 02688	97	15	63325 11891 36789 38604 32759 54593	0.89791
2	102	5 07060 24009 12917 60598 68128 21504	83	16	5 07282 02989 53863 75247 83563 99681	0.89060
13	28	15 50293 28026 62396 21526 95351 05521	89	16	15 49673 14251 78936 43509 93277 30561	0.89106

Table II. (Theorem 5.2(b)).

p_1	x_1	$p_1^{x_1}$	p_2	x_2	$p_2^{x_2}$	delta
2	3	8	3	2	9	.00000
3	3	27	5	2	25	0.21534
2	5	32	3	3	27	0.48832
2	5	32	6	2	36	0.40000
5	3	125	11	2	121	0.28906
2	7	128	11	2	121	0.40575
2	7	128	5	3	125	0.22754
6	3	216	15	2	225	0.40876
2	8	256	3	5	243	0.46694
7	3	343	19	2	361	0.49512
2	9	512	23	2	529	0.45416
2	10	1024	10	3	1000	0.46007
6	4	1296	11	3	1331	0.49607
12	3	1728	42	2	1764	0.48070
2	11	2048	45	2	2025	0.41184
3	7	2187	13	3	2197	0.29941
3	7	2187	47	2	2209	0.40194
13	3	2197	47	2	2209	0.32293
15	4	50625	37	3	50653	0.30762
6	7	2 79936	23	4	2 79841	0.36309
2	50	1 12589 99068 42624	47	9	1 11913 04731 02767	0.85259
2	50	1 12589 99068 42624	18	12	1 15683 13814 26176	0.89628
24	11	1 52168 11431 69024	33	10	1 53157 89852 64449	0.85597
15	13	1 94619 50683 59375	50	9	1 95312 50000 00000	0.83986
2	51	2 25179 98136 85248	19	12	2 21331 49190 66161	0.88532
6	20	3 65615 84400 62976	26	11	3 67034 44869 87776	0.84507
11	15	4 17724 81694 15651	20	12	4 09600 00000 00000	0.89095
28	11	8 29350 94674 71872	39	10	8 14040 60851 91601	0.89154
10	16	10 00000 00000 00000	17	13	9 90457 80329 05937	0.87396
5	23	11 92092 89550 78125	29	11	12 20050 97657 05829	0.89862
2	54	18 01439 85094 81984	30	11	17 71470 00000 00000	0.89096
23	12	21 91462 44320 20321	43	10	21 61148 23132 84249	0.88656
6	21	21 93695 06403 77856	23	12	21 91462 44320 20321	0.81690
6	21	21 93695 06403 77856	43	10	21 61148 23132 84249	0.88845
2	55	36 02879 70189 63968	24	12	36 52034 74360 56576	0.88735

19	13	42 05298 34622 57059	46	10	42 42074 74827 76576	0.87619
3	35	50 03154 50989 99707	33	11	50 54210 65137 26817	0.88076
13	15	51 18589 30140 90757	33	11	50 54210 65137 26817	0.88656
26	12	95 42895 66616 82176	35	11	96 54915 73730 46875	0.88631
35	11	96 54915 73730 46875	50	10	97 65625 00000 00000	0.88575
14	15	155 56809 55578 12224	21	13	154 47237 77391 19461	0.87497
11	17	505 44702 84992 93771	23	13	504 03636 19364 67383	0.85578
7	21	558 54586 40832 84007	41	11	550 32903 17162 48441	0.89708
6	23	789 73022 30536 02816	31	12	787 66278 37885 49761	0.85579
6	23	789 73022 30536 02816	19	14	799 00668 57828 84121	0.89216
19	14	799 00668 57828 84121	31	12	787 66278 37885 49761	0.89710
26	13	2481 15287 32037 36576	47	11	2472 15921 50840 12303	0.86739
28	13	6502 11142 24979 47648	37	12	6582 95200 58400 35281	0.89872
15	16	6568 40835 57128 90625	28	13	6502 11142 24979 47648	0.89414
15	16	6568 40835 57128 90625	37	12	6582 95200 58400 35281	0.85892
2	65	36893 48814 74191 03232	5	28	37252 90298 46191 40625	0.89721
37	13	2 43569 22421 60813 05397	50	12	2 44140 62500 00000 00000	0.87101
2	68	2 95147 90517 93528 25856	29	14	2 97558 23267 57994 63481	0.89800
11	20	6 72749 99493 25600 09201	40	13	6 71088 64000 00000 00000	0.87486
5	30	9 31322 57461 54785 15625	41	13	9 25103 10231 50136 29321	0.89638
35	14	41 39545 12236 93847 65625	46	13	41 29065 87698 35408 01536	0.87993
19	17	54 80386 85778 48021 85939	47	13	54 60990 70612 05831 77327	0.88730
6	28	61 40942 21446 48154 97216	23	16	61 32610 41568 09986 48961	0.86842
2	73	94 44732 96573 92904 27392	7	26	93 87480 33764 77543 05649	0.89920
20	17	131 07200 00000 00000 00000	38	14	130 90925 53986 67734 38464	0.86863
2	74	188 89465 93147 85808 54784	39	14	188 32349 19413 17426 09041	0.88695
2	75	377 78931 86295 71617 09568	41	14	379 29227 19491 55588 02161	0.89368
3	49	2392 99329 23061 75295 90083	17	19	2390 72435 68515 13248 47153	0.87071
15	20	3325 25673 00796 50878 90625	37	15	3334 46267 95181 53070 88493	0.89126
19	19	19784 19655 66031 35891 23979	33	16	19779 85201 46255 88779 34081	0.84943
7	29	32199 05755 81317 97268 37607	13	22	32118 38877 95485 51051 57369	0.89390
2	84	1 93428 13113 83406 67952 98816	3	53	1 93832 45667 68001 98967 96723	0.89402
7	30	2 25393 40290 69225 80878 63249	31	17	2 25501 16774 16274 31786 82911	0.86903
11	25	10 83470 59433 88372 20418 30251	34	17	10 84280 35605 96593 23542 07744	0.87991
14	23	22 95856 92886 98149 54822 20544	18	21	22 94682 51895 12940 71398 72768	0.87516
23	20	171 61558 31334 58634 29238 95201	40	17	171 79869 18400 00000 00000 00000	0.89088
6	35	171 90707 99748 42259 10286 58176	23	20	171 61558 31334 58634 29238 95201	0.89829
6	35	171 90707 99748 42259 10286 58176	40	17	171 79869 18400 00000 00000 00000	0.88250
15	25	25251 16829 40423 48861 69433 59375	43	18	25259 93335 73498 06081 18208 06649	0.88234