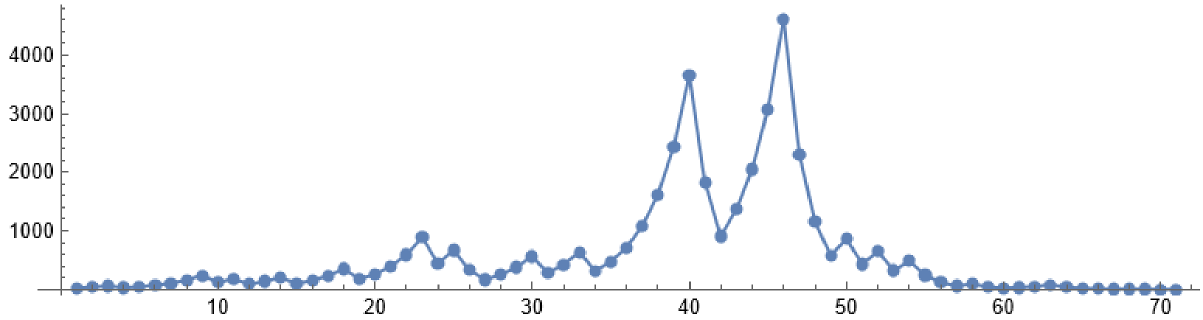

The $3n + 1$ Conjecture

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The $3n + 1$ process

Take a natural number.

- If it is even, then divide it by 2.
- If it is odd, then multiply it by 3 and add 1.

Iterate this, until you're tired.

$1 \to 4 \to 2 \to 1 \to 4 \to 2 \to 1 \to \ldots$ (I'm tired already)
$2 \to$ (seen that one) $\ldots \to 1$
$3 \to 10 \to 5 \to 16 \to 8 \to 4 \to$ (seen that one) $\ldots \to 1$
$4 \to$ (seen that one) $\ldots \to 1$
$5 \to$ (seen that one) $\ldots \to 1$
$6 \to 3 \to$ (seen that one) $\ldots \to 1$
$7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10$ (seen that one) $\ldots \to 1$

The $3n + 1$ function

After an odd number *always* an even number appears, so *always* a division by 2 follows. Those two steps we take together as one step.

The $3n + 1$ function $T : \mathbb{N} \to \mathbb{N}$ is defined by

$$T(n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (3n+1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

The $3n + 1$ process *iterates* T:

$n \to T(n) \to T^2(n) \to T^3(n) \to \ldots \to T^k(n) \to \ldots$

Example:

$7 \to 11 \to 17 \to 26 \to 13 \to 20 \to 10 \to 5 \to 8 \to 4 \to 2 \leftrightarrows 1$

The $3n + 1$ Conjecture

The $3n + 1$ Conjecture is:

For every natural number n as initial value the $3n + 1$ process will reach 1, and ends in $\ldots \to 2 \leftrightarrows 1$:

$n \to T(n) \to T^2(n) \to \ldots \to 2 \leftrightarrows 1.$

More formal:

For every $n \in \mathbb{N}$ there exist a $k \in \mathbb{N}$ such that $T^k(n) = 1$.

The cycle $1 \leftrightarrows 2$ is called the *trivial cycle*.

Definitions

For $n \in \mathbb{N}$ the sequence $n \to T(n) \to T^2(n) \to \dots$ is called its *orbit*.

If for some $k, n \in \mathbb{N}$ it happens that $T^k(n) = n$
then $\left(n, T(n), T^2(n), \dots, T^{k-1}(n)\right)$ is called a *cycle*.

E.g. the trivial cycle is denoted by $(1, 2)$ (or $(2, 1)$).

If an orbit contains a cycle then it ends there, and the orbit is called *convergent*.
An orbit that is not convergent is called *divergent*.

A divergent orbit is unbounded.

Some convergent orbits seem divergent for a long time:

$27 \to 41 \to 62 \to 31 \to 47 \to 71 \to 107 \to 161 \to 242 \to 121 \to 182 \to 91 \to 137 \to 206 \to 103 \to 155 \to 233 \to 350 \to 175 \to 263 \to 395 \to 593 \to 890 \to 445 \to 668 \to 334 \to 167 \to 251 \to 377 \to 566 \to 283 \to 425 \to 638 \to 319 \to 479 \to 719 \to 1079 \to 1619 \to 2429 \to 3644 \to 1822 \to 911 \to 1367 \to 2051 \to 3077 \to 4616 \to 2308 \to 1154 \to 577 \to 866 \to 433 \to 650 \to 325 \to 488 \to 244 \to 122 \to 61 \to 92 \to 46 \to 23 \to 35 \to 53 \to 80 \to 40 \to 20 \to 10 \to 5 \to 8 \to 4 \to 2 \leftrightarrows 1$

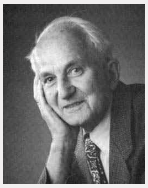
Partial conjectures

The $3n + 1$ Conjecture splits up in two independent conjectures:

The *Cycle Conjecture*:
The trivial cycle is the only cycle for the $3n + 1$ function.

The *Convergence Conjecture*:
There are no divergent orbits for the $3n + 1$ function.

History



Lothar Collatz (1910–1990, Hamburg) said he invented it around 1930.

In 1950 at the ICM in Cambridge (Mass.) he discussed it with some colleagues.

In the 60s first papers appear on variants (a.o. a variant described by poet Raymond Queneau, related to rhyme schemes in 12th century poetry).

In 1971 the $3n + 1$ problem of (dis)proving the conjecture appears in print. Martin Gardner writes about it in 1972, and then it goes viral.

It is known by many names: Collatz Conjecture, Syracuse problem, problem of Hasse, of Kakutani, of Coxeter, of Ulam, hailstone number problem, etc.

The 60s: 8 papers, the 70s: 34 papers, the 80s: 52 papers, the 90s: 103 papers, the 00s: 134 papers.

Survey Literature

- Jeffrey C. Lagarias (ed.): *The Ultimate Challenge: The* $3x + 1$ *Problem*, Am. Math. Soc., 2010 (a collection of important papers)
- Jeffrey C. Lagarias, “The $3x + 1$ problem and its generalizations”, Am. Math. Monthly **92** [1985], 3–23 (detailed history until 1985)
- Jeffrey C. Lagarias, “The $3x + 1$ problem: an annotated bibliography (1963–1999)”, arxiv.org. (short summaries of all papers until 2000)
- Jeffrey C. Lagarias, “The $3x + 1$ problem: an annotated bibliography II (2000–2009)”, arxiv.org. (short summaries of all papers in 2000–2009)

Why the $3n + 1$ Conjecture may be true

Arguments / heuristics:

- Experimental
- Probabilistic
- Logic / Complexity theoretic
- Diophantine
- Dynamical systems / ergodic
- Terence Tao

Reformulations:

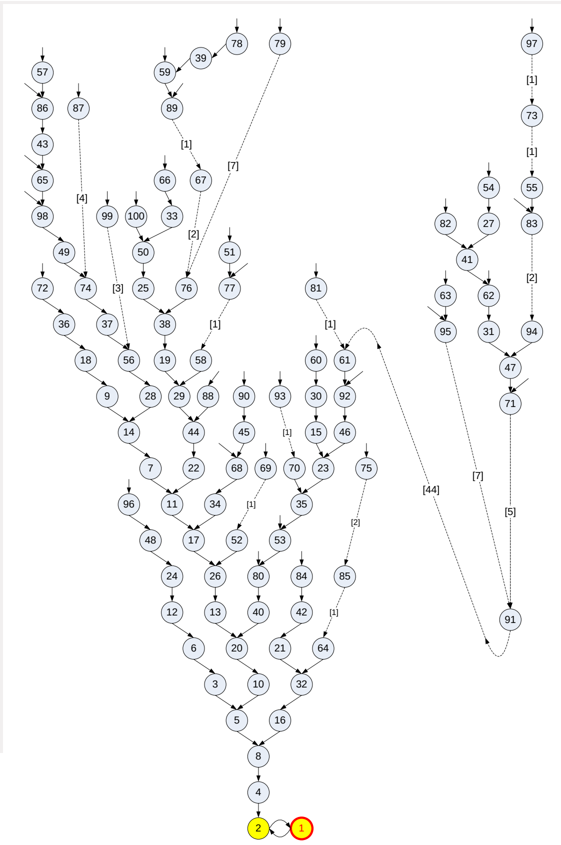
- Graphs
- Modular and De Bruijn graphs
- Infinite matrices (operators)
- Eigenspaces
- Functional equations

The $3n + 1$ graph

The $3n + 1$ Conjecture is equivalent to:

The $3n + 1$ Graph is connected, i.e. has exactly one connected component.

This is a reformulation that is not very informative on its own.



Graphs for similar functions: $3n-1$

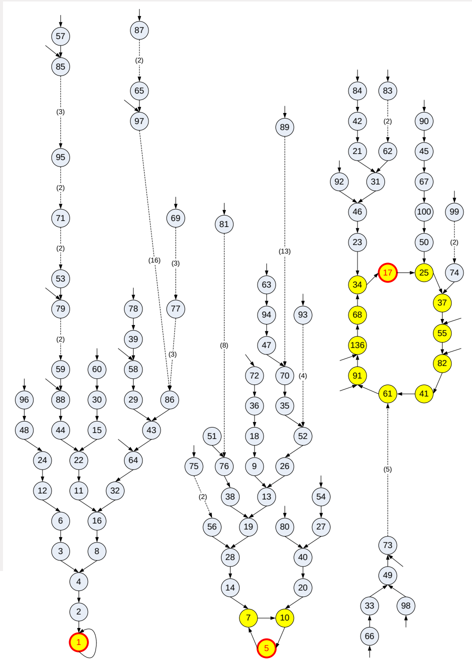
The $3n-1$ function:

$$T_{3,-1}(n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (3n-1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

Equivalent to $3n+1$ on the negative integers:
$T_{3,-1}(n) = -T(-n)$

There are (conjectured only) 3 cycles, and (conjectured) no divergent orbits

This graph is not connected, it has (at least) 3 connected components.

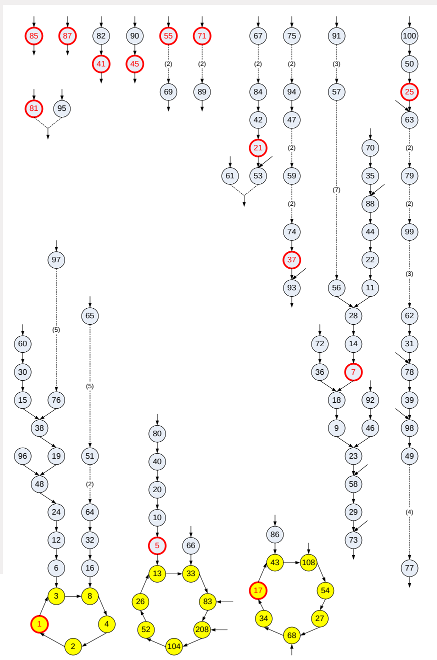


Graphs for similar functions: $5n+1$

The $5n+1$ function:

$$T_{5,1}(n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (5n+1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

There are (conjectured only) 3 cycles, but also (conjectured) infinitely many divergent orbits



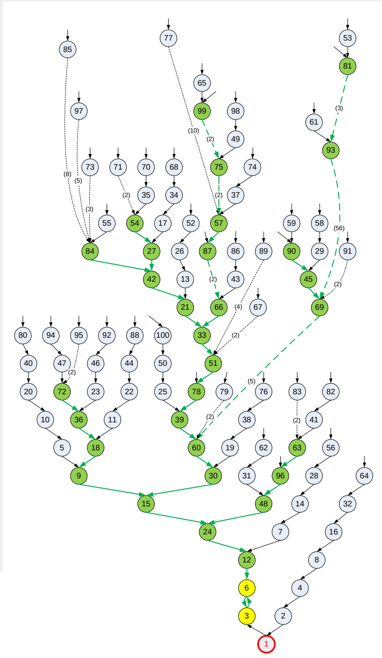
Graphs for similar functions: $3n + 3$

The $3n + 3$ function:

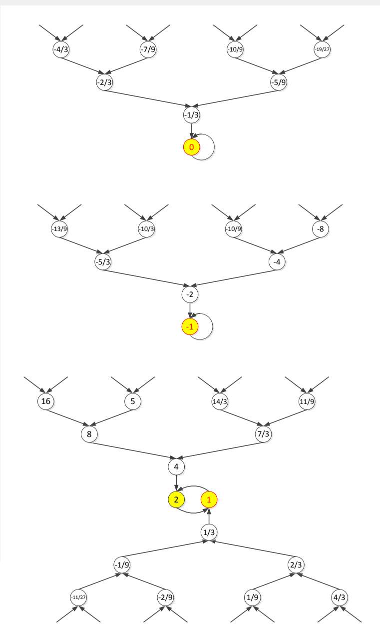
$$T_{3,3}(n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (3n+3)/2 & \text{if } n \text{ is odd.} \end{cases}$$

The subgraph of all multiples of 3 is isomorphic to the $3n + 1$ graph

This graph is again conjectured to be connected (all orbits end in the trivial cycle)



Graphs for similar functions: rational $3n + 1$



$T(n)$ but now for rational n in lowest terms with odd denominator defined by

$$T(n) = \begin{cases} n/2 & \text{if } n \text{ has even numerator,} \\ (3n+1)/2 & \text{if } n \text{ has odd numerator.} \end{cases}$$

(even denominator is uninteresting)

In fact this is the combined $3n + q$ graph for all odd q. Every $\frac{a}{b}$ with odd b has two predecessors: $\frac{2a}{b}$ and $\frac{2a-b}{3b}$

There are (conjectured) no divergent orbits, and there are infinitely many convergent orbits, which allow a very nice description

All cycles in the rational $3n + 1$ graph

A convergent orbit ends in a cycle, a cycle has an even-odd structure:

- length 1: 2 possibilities: **e**, **o**
- length 2: 1 possibility: **eo**
- length 3: 2 possibilities: **eeo**, **eoo**
- length 4: 3 possibilities: **eeeo**, **eeoo**, **eooo**
- length 5: 6 possibilities.: **eeeeo**, **eeeoo**, **eeoeo**, **eeooo**, **eoeoo**, **eoooo**

etcetera, the so called *Lyndon words*.

For every Lyndon word an equation for the cycle can be derived, which has exactly one rational solution:

e.g. for **eoo** we derive $x \xrightarrow{e} \frac{1}{2}x \xrightarrow{o} \frac{3}{4}x + \frac{1}{2} \xrightarrow{o} \boxed{\frac{9}{8}x + \frac{5}{4} = x}$, so $x = -10$, this yields the known cycle $-10 \to -5 \to -7 \to -10$

and from the cycle the complete connected component can easily be computed backwards

Conway's "amusical permutation"

$$U(n) = \begin{cases} (3n)/2 & \text{if } n \text{ is even,} \\ (3n+1)/4 & \text{if } n \equiv 1 \pmod 4, \\ (3n-1)/4 & \text{if } n \equiv -1 \pmod 4 \end{cases}$$

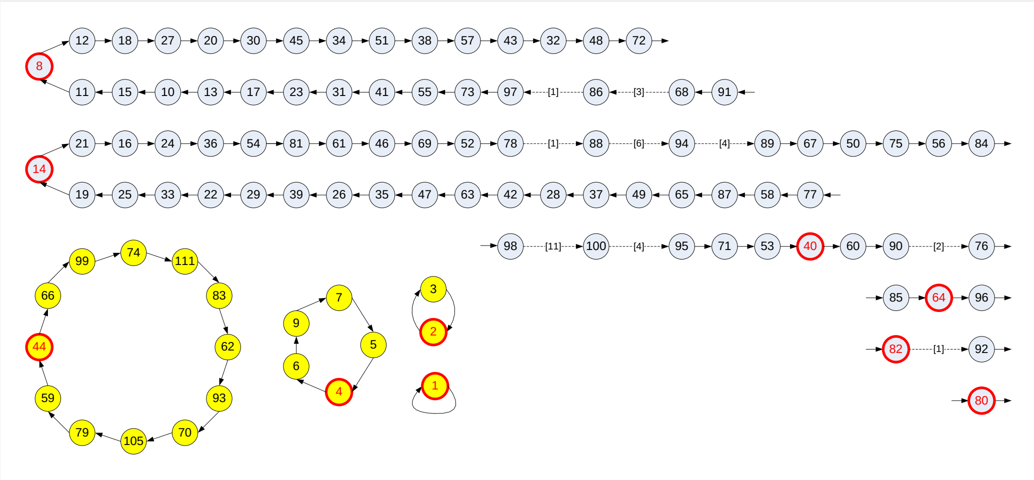
now again on the integers

Seems to be older than the $3n + 1$ function T itself (says Collatz).
This function is a permutation on $\mathbb{N}$. So U^{-1} exists:

$$U^{-1}(n) = \begin{cases} (2n)/3 & \text{if } n \equiv 0 \pmod 3, \\ (4n-1)/3 & \text{if } n \equiv 1 \pmod 3, \\ (4n+1)/3 & \text{if } n \equiv -1 \pmod 3 \end{cases}$$

This gives a linear graph: every node has one predecessor and one successor.

The ‘amusical graph’



There are probably no other cycles, and infinitely many divergent connected components.

Experimental

Eric Roosendaal, Tomás Oliveira e Silva, David Barina, and others:

The $3n+1$ Conjecture holds for all $n < 2^{69} \approx 5.9 \times 10^{20}$.

This is an impressive achievement. Distributed computations are ongoing. It’s intelligent brute force computing, with e.g.

- stop as soon as $T^k(n) \leq n$,
- performing multiple steps at once with a time-memory tradeoff.

See Eric Roosendaal’s website `http://www.ericr.nl/wondrous/`.

Probabilistic

The T function with inputs n from a uniformly random even/odd distribution produces also uniformly random even/odd distributed outputs $T(n)$.

So $T(n) \approx \frac{3}{2}n$ with probability $1/2$, and $T(n) = \frac{1}{2}n$ with probability $1/2$, so

$$T^k(n) \approx \left(\frac{3}{2}\right)^{k/2} \left(\frac{1}{2}\right)^{k/2} n = \left(\frac{1}{2}\sqrt{3}\right)^k n.$$

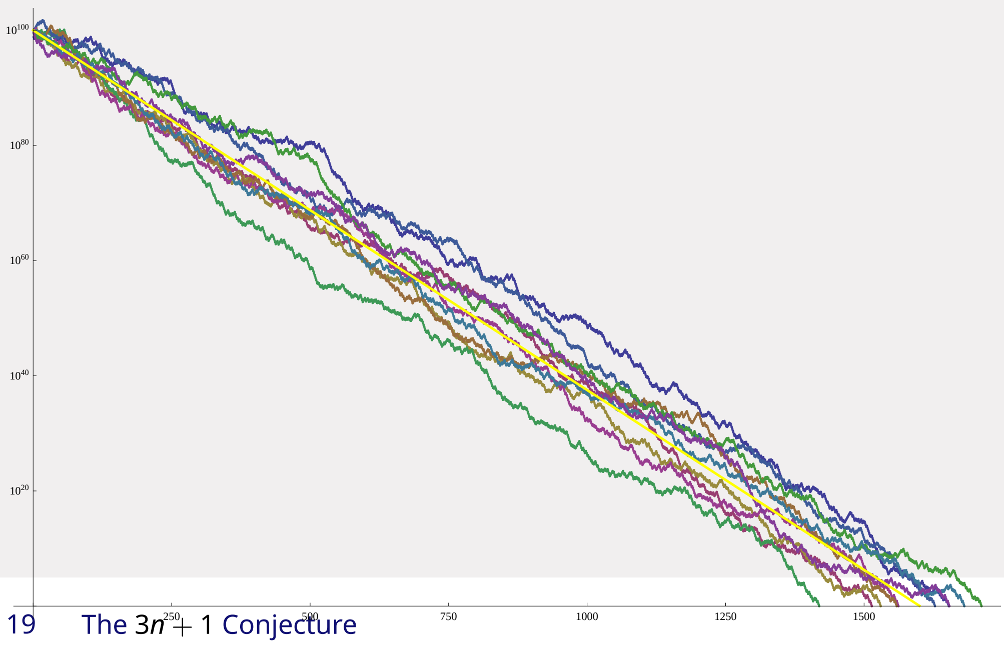
Note that $\frac{1}{2}\sqrt{3} \approx 0.866 < 1$. After $k \approx 6.95 \log n$ steps one expects to hit 1.

Similarly for $3n + q$ with any q, as well as for rational $3n + 1$.

More advanced stochastic models predict things like:

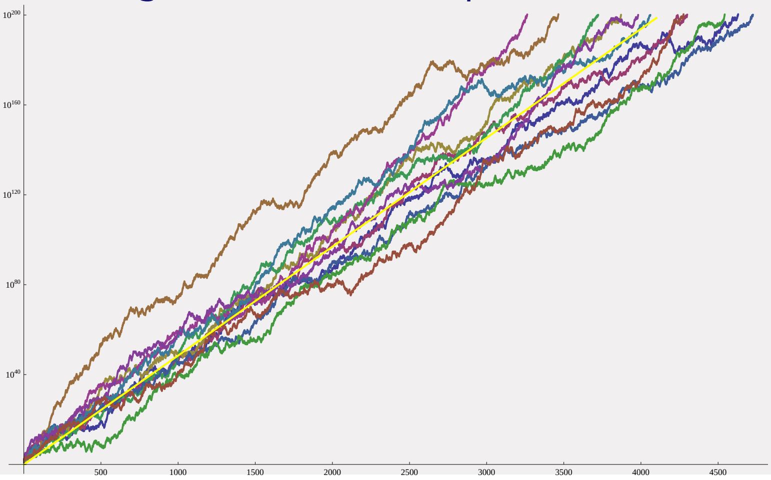
- extreme high orbits reach highest point $\approx n^2$ after $\approx 7.65 \log n$ steps, and then need another $\approx 13.9 \log n$ steps to reach 1,
- extreme long orbits will reach length $\approx 41.7 \log n$.

Experiment for $3n + 1$



Experiment for $5n+1$

Argument also works for $5n+1$: factor now is $\frac{1}{2}\sqrt{5} \approx 1.12$, so divergence now is most probable.



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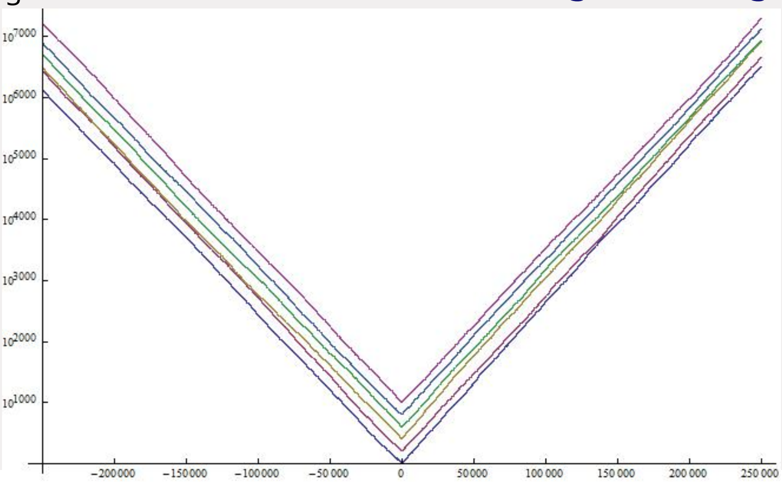
Experiment for the amusical permutation

For the amusical permutation the factor is $\frac{3}{4}\sqrt{2} \approx 1.061$ forwards, and $\frac{2}{3}\sqrt[3]{4} \approx 1.058$ afterwards, so divergence is highly probable in both directions.



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Logic en Complexity theoretic

Conway: there exists a generalisation of the $3n+1$ function whose iteration simulates a universal computer (Turing machine).

For this function the decision problem "does an orbit reach a random power of 2" computationally undecidable.

In his paper "On unsettleable arithmetical problems" (Am. Math. Monthly **120** [March 2013], 192–198) John Conway says:

> "It is likely that some simple Collatzian problems (possibly even the $3n+1$ problem itself) will remain forever unsettleable."

I say: don't let that discourage you...

Diophantine

> An m-cycle is a cycle for the $3n+1$ function with m local maxima and m local minima.

Ray Steiner proved (in 1977) that the trivial cycle is the only 1-cycle. The argument roughly is as follows:

If, starting from n, you first do k upward (odd) steps ($T(n) = (3n+1)/2$) and then arrive at an even number, then $n = a2^k - 1$ with odd a, and $T^k(n) = a3^k - 1$.

Then follow ℓ downward (even) steps ($T(n) = n/2$), and you return at n, so $a3^k - 1 = 2^\ell \left(a2^k - 1\right)$, so

$$0 < 2^{k+\ell} - 3^k = \frac{2^\ell - 1}{a} < 2^\ell$$

Diophantine (continued)

Take logarithms: $0 < 1 - \frac{3^k}{2^{k+\ell}} < \frac{1}{2^k} \Rightarrow |(k+\ell)\log 2 - k\log 3| < \frac{1}{2^k}$.
The "Theory of Linear Forms in Logarithms of Algberaic Numbers" is a branch of Number Theory that says (a.o.) that powers of integers cannot be close to each other. In particular (Alan Baker (1966), Georges Rhin (1987)):

$$|(k+\ell)\log 2 - k\log 3| > k^{-13.3}.$$

Comparing the bounds gives $k \leq 85$. Steiner's upper bound was $k < 10^{200}$.
With computing the "continued fraction" of $\frac{\log 3}{\log 2}$ to high precision, a reduced upper bound can be found.

Diophantine (continued)

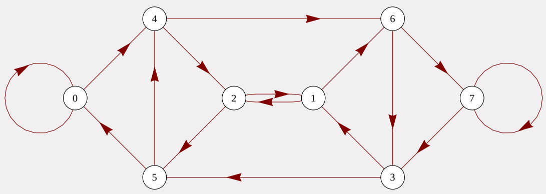
This argument by Steiner can be generalised.
John Simons (Groningen) proved in 2004 that 2-cycles do not exist.
Simons & dW then proved in 2005-2022, based on the lower bound 2^{69} for a starting value:

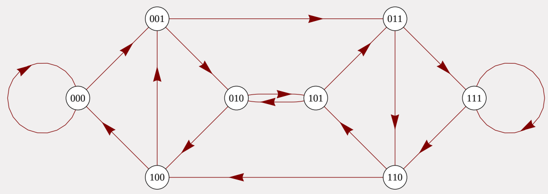
m-cycles with $2 \leq m \leq 77$ do not exist.
If a nontrivial cycle exists, then its length exceeds 10^{10}.

Christian Hercher (2021) claims to have proved that there are no m-cycles with $m \leq 90$ but I have not checked his proof (yet).

Modular $3n+1$ graphs and De Bruijn graphs

The modular $3n+1$ graph with modulus m consists of all possible arrows from $n \pmod m$ to $T(n) \pmod m$.





(Laarhoven-dW 2013) These graphs turn out to have a beautiful structure when $m = 2^k$: then they are "De Bruijn graphs" of order k (after our own N.G. de Bruijn), coming from the bit-shift operator $a_1a_2\cdots a_k \to \begin{cases} a_2\cdots a_k0 \\ a_2\cdots a_k1 \end{cases}$.

Conjugation

For every $pn+q$ function (p, q odd, coprime) the modular graph with modulus 2^k is the same kth order De Bruijn graph. Only the labeling of the nodes is different for each p, q.

This labeling is the conjugation map Φ_k:

$$\Phi_k : n \to a_1a_2\cdots a_k, \text{ where } a_i \text{ is the parity (even or odd) of } T^{i-1}(n).$$

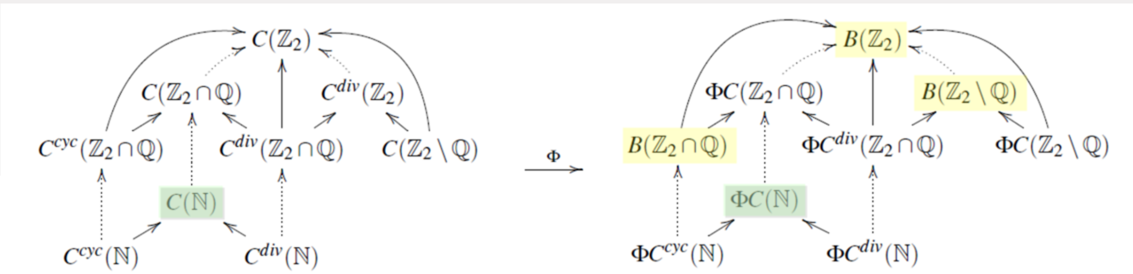
The $pn+q$ graph modulo 2^k itself is independent of p, q, but the conjugation map does depend on p, q.

The infinite De Bruijn graph

For $k \to \infty$ one gets the infinite De Bruijn graph $B(\mathbb{Z}_2)$, based on the bit-shift shift operator on infinite bit sequences: $a_1 a_2 a_3 \cdots \to a_2 a_3 \cdots$.

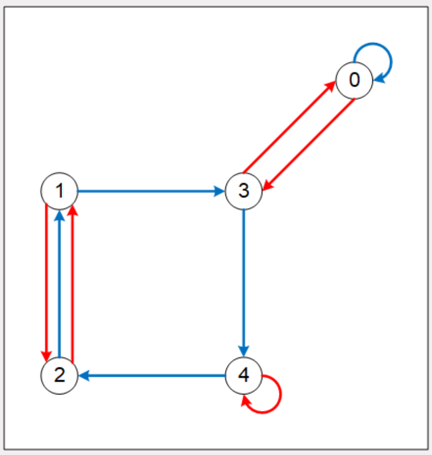
This graph contains all possible cycles (from the Lyndon words), each node decorated with a full binary tree, and uncountably many connected components without cycle, each being a binary tree extended infinitely to both sides.

This graph is a real monstrosity, loaded with structure. Hiding in there are all possible $pn + q$-graphs, identified by their (weird) conjugacy maps.



Modular $3n+1$ graphs for odd moduli m coprime to 3

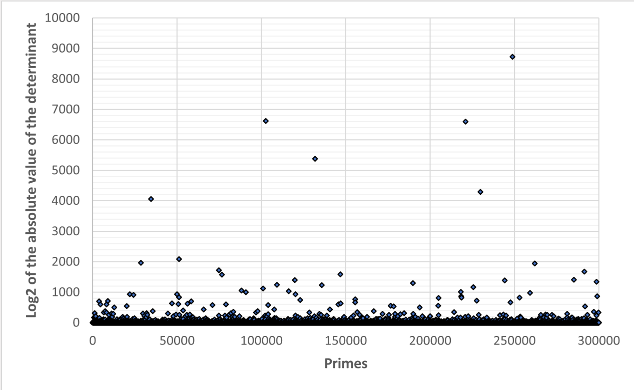
Structure for odd moduli is more intricate: following always the path $n \to \dfrac{3n+1}{2} \pmod{m}$ gives a set of cycles, following always the path $n \to \dfrac{n}{2} \pmod{m}$ gives a different set of cycles, e.g. for $m = 5$: red arrows: $n \to \dfrac{3n+1}{2} \pmod{5}$, blue arrows: $n \to \dfrac{n}{2} \pmod{5}$



Modular $3n+1$ graphs for odd moduli m coprime to 3

Adjacency matrix $A = (a_{i,j})$ of a graph:
$a_{n,T(n)}$ = number of arrows from n to $T(n)$
e.g. for the $3n+1$ graph $\pmod 5$:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$



Achilleas Karras experimentally found wild results for determinants of such matrices

we found a formula explaining this (ongoing research, Karras & dW)

Infinite matrices

The $3n+1$ graph is infinite, so it has an infinite adjacency matrix $A = (a_{i,j})$ with $a_{n,T(n)} = 1$ for all $n \in \mathbb{N}$, and $a_{i,j} = 0$ otherwise. This graph is very structured:

$$A = \begin{pmatrix}
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \ddots \\
 & & & & \vdots & & & & & & & & & & \\
 & & & & & \vdots & & & & & & & & &
\end{pmatrix}$$

Powers of adjacency matrices

The kth power of A describes paths of length k. For growing k only the first two columns seem to survive:

$$A^{2k} = \begin{pmatrix} 1 & \cdot & \cdots \\ \cdot & 1 & \cdots \\ \cdot & 1 & \cdots \\ 1 & \cdot & \cdots \\ 1 & \cdot & \cdots \\ 1 & \cdot & \cdots \\ \cdot & 1 & \cdots \\ \cdot & 1 & \cdots \\ \cdot & 1 & \cdots \\ \cdot & 1 & \cdots \\ 1 & \cdot & \cdots \\ \cdot & 1 & \cdots \\ & \vdots & \end{pmatrix}, A^{2k+1} = \begin{pmatrix} \cdot & 1 & \cdots \\ 1 & \cdot & \cdots \\ 1 & \cdot & \cdots \\ \cdot & 1 & \cdots \\ \cdot & 1 & \cdots \\ \cdot & 1 & \cdots \\ 1 & \cdot & \cdots \\ 1 & \cdot & \cdots \\ 1 & \cdot & \cdots \\ 1 & \cdot & \cdots \\ \cdot & 1 & \cdots \\ 1 & \cdot & \cdots \\ & \vdots & \end{pmatrix}$$

The $3n+1$ Conjecture is equivalent to the statement that for $k \to \infty$ the matrices A^{2k} and A^{2k+1} converge to resp. $(a_0, a_1, 0, 0, \ldots)$ and $(a_1, a_0, 0, 0, \ldots)$, where $a_0 + a_1 = (1, 1, 1 \ldots)^\top$.

Eigenspaces

Note that $Aa_0 = a_1$ and $Aa_1 = a_0$, so $a_0 + a_1$ is an eigenvector for the eigenvalue 1, and $a_0 - a_1$ is an eigenvector for the eigenvalue -1.

In general for such graphs: a cycle of length k is associated to k eigenvectors, for the kth roots of unity as eigenvalues.

One of those eigenvalues is 1. Indeed, each connected component, convergent or divergent, corresponds to an eigenvector for the eigenvalue 1: the vector with 1s on the connected component's elements.

The complete spectra of A and $A^\top$ have been described (dW, ongoing work).

(Engl, 1982) The dimension of the full eigenspace of A for the eigenvalue 1 equals the number of connected components of the graph.

So the $3n+1$ Conjecture is equivalent with the statement that the eigenspace for the eigenvalue 1 of the matrix A is one-dimensional.

Functional equations

Berg and Meinardus (1994) consider the power series

$$f(z) = e_0 + e_1 z + e_2 z^2 + \ldots + e_n z^n + \ldots,$$

where we demand that $e_n = e_{T(n)}$ for all n. What can we say about f? Define

$$f_k(z) = \sum_{n=0}^{\infty} e_{3n+k} z^{3n+k} \text{ for } k = 0, 1, 2, \text{ so } f = f_0 + f_1 + f_2.$$

Let $\omega = e^{2\pi i/3}$. Then $f_0(\omega z) = f_0(z), \quad f_1(\omega z) = \omega f_1(z), \quad f_2(\omega z) = \overline{\omega} f_2(z)$.
And we find $f_2(z) = \frac{1}{3}\left(f(z) + \omega f(\omega z) + \overline{\omega} f(\overline{\omega} z)\right)$.

Functional equations (continued)

On the other hand

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} e_{2n} z^{2n} + \sum_{n=0}^{\infty} e_{2n+1} z^{2n+1} = \sum_{n=0}^{\infty} e_{T(2n)} z^{2n} + \sum_{n=0}^{\infty} e_{T(2n+1)} z^{2n+1} \\ &= \sum_{n=0}^{\infty} e_n z^{2n} + \sum_{n=0}^{\infty} e_{3n+2} z^{2n+1} = f(z^2) + z^{-1/3} f_2(z^{2/3}). \end{aligned}$$

So f satisfies the functional equation

$$3z\left(f(z^3) - f(z^6)\right) = f(z^2) + \omega f(\omega z^2) + \overline{\omega} f(\overline{\omega} z^2)$$

where $\omega = -\frac{1}{2} + \frac{1}{2} i\sqrt{3}$.

Functional equations (continued)

This functional equation

$$3z\left(f(z^3) - f(z^6)\right) = f(z^2) + \omega f(\omega z^2) + \overline{\omega} f(\overline{\omega} z^2)$$

is linear. Two obvious solutions are $f(z) = 1$ and $f(z) = \dfrac{z}{1-z} = z + z^2 + z^3 + \ldots$.

Again there is a direct correspondence between independent solutions of this functional equation and the connected components of the $3n+1$ graph.

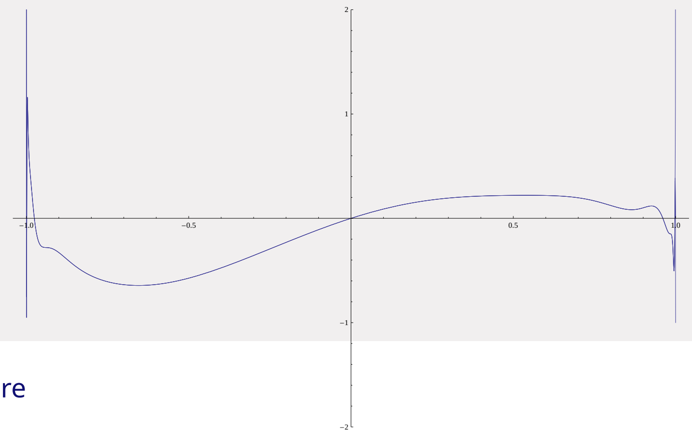
The $3n+1$ Conjecture is equivalent to the statement that this functional equation has a two-dimensional solution space in power series.

Functional equations (continued)

With $e_n = -e_{T(n)}$ we find a more interesting functional equation:

$$3z\left(-f(z^3) - f(z^6)\right) = f(z^2) + \omega f(\omega z^2) + \overline{\omega} f(\overline{\omega} z^2)$$

with as solution $f(z) = z - z^2 - z^3 + z^4 + z^5 + z^6 - z^7 - \ldots$. Here $e_n = \pm 1$ indicates an even or odd number of steps from n to 1.



The amusical functional equation

Doing the same for Conway's amusical permutation gives (with $i = \sqrt{-1}$)

$$2z\left(2f(z^4) - f(z^6) - f(-z^6)\right) = \\ = (z^2+1)\left(f(z^3) - f(-z^3)\right) - i(z^2-1)\left(f(iz^3) - f(-iz^3)\right)$$

The four cycles in the graph are finite connected components, implying that this functional equation allows polynomial solutions (next to $f(z) = 1$):
$f(z) = z, \quad f(z) = z^2 + z^3, \quad f(z) = z^4 + z^5 + z^6 + z^7 + z^9$ and

$$f(z) = z^{44} + z^{59} + z^{62} + z^{66} + z^{70} + z^{74} + z^{79} + z^{83} + z^{93} + z^{99} + z^{105} + z^{111}.$$

Probably there are no more independent polynomial solutions. And there are (probably infinitely many) power series solutions, such as
$f(z) = z^8 + z^{10} + z^{11} + z^{12} + z^{13} + z^{15} + z^{17} + z^{18} + z^{20} + \ldots$.

Terence Tao

Study behaviour of "almost all" orbits, i.e. disregard "outliers"

Terras (1976): For "almost all" n there exists a k such that $T^k(n) < n$.

("almost all" means: with "density" 1, for some useful definition of "density")

Tao (2019): For "almost all" n there exists a k such that $T^k(n) < f(n)$, for *any* function f that grows to ∞, e.g. $f(n) = \log\log\log\log n$.

Terence Tao, *Almost all orbits of the Collatz map attain almost bounded values*, `https://arxiv.org/abs/1909.03562`
Pretty difficult paper, too hard for me, uses a.o. transcendence theory and ergodic theory.

Conclusion

Paul Erdős: “Mathematics is not yet ripe enough for such questions.”

Ionica Smeets: “The $3n+1$ Conjecture is an “embarrassing problem”: easily stated, but it is hard to explain to laymen why us brilliant mathematicians can’t solve it.”

Terence Tao: “The Collatz Conjecture is one of the most “dangerous” conjectures known — notorious for absorbing massive amounts of time from both professional and amateur mathematicians.”

BdW: Exploring the relations with many different areas of mathematics gives some insight in why the problem is difficult, and some hope that in one of those areas something of interest is hiding.

http://xkcd.com/710/

Questions?